Appendix D Selected Solutions

Chapter 1

1. Let $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, -4)$. Perform the following computations and draw the vectors relative to a 2D coordinate system.

a) $\mathbf{u} + \mathbf{v}$
b) $\mathbf{u} - \mathbf{v}$
c) $2\mathbf{u} + \frac{1}{2}\mathbf{v}$
d) $-2\mathbf{u} + \mathbf{v}$

Solution:

a) $(1, 2) + (3, -4) = \big(1 + 3, 2 + (-4)\big) = (4, -2)$
b) $(1, 2) - (3, -4) = (1, 2) + (-3, 4) = (1 - 3, 2 + 4) = (-2, 6)$
c) $2(1, 2) + \frac{1}{2}(3, -4) = (2, 4) + \left(\frac{3}{2}, -2\right) = \left(\frac{7}{2}, 2\right)$
d) $-2(1, 2) + (3, -4) = (-2, -4) + (3, -4) = (1, -8)$

3. This exercise shows that vector algebra shares many of the nice properties of real numbers (this is not an exhaustive list). Assume $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$. Also assume that c and k are scalars. Prove the following vector properties.

a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative Property of Addition)
b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (Associative Property of Addition)
c) $(ck)\mathbf{u} = c(k\mathbf{u})$ (Associative Property of Scalar Multiplication)
d) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (Distributive Property 1)
e) $\mathbf{u}(k + c) = k\mathbf{u} + c\mathbf{u}$ (Distributive Property 2)

Solution:

a)

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_x, u_y, u_z) + (v_x, v_y, v_z) \\
&= (u_x + v_x, u_y + v_y, u_z + v_z) \\
&= (v_x + u_x, v_y + u_y, v_z + u_z) \\
&= (v_x, v_y, v_z) + (u_x, u_y, u_z) \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

b)

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (u_x, u_y, u_z) + \big((v_x, v_y, v_z) + (w_x, w_y, w_z)\big)$$

$$
\begin{aligned}
&= (u_x, u_y, u_z) + (v_x + w_x, v_y + w_y, v_z + w_z) \\
&= \left(u_x + (v_x + w_x), u_y + (v_y + w_y), u_z + (v_z + w_z)\right) \\
&= \left((u_x + v_x) + w_x, (u_y + v_y) + w_y, (u_z + v_z) + w_z\right) \\
&= (u_x + v_x, u_y + v_y, u_z + v_z) + (w_x, w_y, w_z) \\
&= \left((u_x, u_y, u_z) + (v_x, v_y, v_z)\right) + (w_x, w_y, w_z) \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}
$$

c)

$$
\begin{aligned}
(ck)\mathbf{u} &= (ck)(u_x, u_y, u_z) \\
&= \left((ck)u_x, (ck)u_y, (ck)u_z\right) \\
&= \left(c(ku_x), c(ku_y), c(ku_z)\right) \\
&= c(ku_x, ku_y, ku_z) \\
&= c(k\mathbf{u})
\end{aligned}
$$

d)

$$
\begin{aligned}
k(\mathbf{u} + \mathbf{v}) &= k\left((u_x, u_y, u_z) + (v_x, v_y, v_z)\right) \\
&= k(u_x + v_x, u_y + v_y, u_z + v_z) \\
&= \left(k(u_x + v_x), k(u_y + v_y), k(u_z + v_z)\right) \\
&= (ku_x + kv_x, ku_y + kv_y, ku_z + kv_z) \\
&= (ku_x, ku_y, ku_z) + (kv_x, kv_y, kv_z) \\
&= k\mathbf{u} + k\mathbf{v}
\end{aligned}
$$

e)

$$
\begin{aligned}
\mathbf{u}(k + c) &= (u_x, u_y, u_z)(k + c) \\
&= \left(u_x(k + c), u_y(k + c), u_z(k + c)\right) \\
&= (ku_x + cu_x, ku_y + cu_y, ku_z + cu_z) \\
&= (ku_x, ku_y, ku_z) + (cu_x, cu_y, cu_z) \\
&= k\mathbf{u} + c\mathbf{u}
\end{aligned}
$$

5. Let $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{v} = (3, -4, 1)$. Normalize $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \left(-\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

$$\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2 + 1^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{3}{\sqrt{26}}, -\frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)$$

7. Is the angle between $\mathbf{u}$ and $\mathbf{v}$ orthogonal, acute, or obtuse?

a) $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (2, 3, 4)$
b) $\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (-2, 2, 0)$
c) $\mathbf{u} = (-1, -1, -1)$, $\mathbf{v} = (3, 1, 0)$

a) $\mathbf{u} \cdot \mathbf{v} = 1(2) + 1(3) + 1(4) = 9 > 0 \Rightarrow$ acute
b) $\mathbf{u} \cdot \mathbf{v} = 1(-2) + 1(2) + 0(0) = 0 \Rightarrow$ orthogonal
c) $\mathbf{u} \cdot \mathbf{v} = -1(3) + (-1)(1) + (-1)(0) = -4 < 0 \Rightarrow$ obtuse

9. Let $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$. Also let c and k be scalars. Prove the following dot product properties.

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
d) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
e) $\mathbf{0} \cdot \mathbf{v} = 0$

Solution:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_x, u_y, u_z) \cdot (v_x, v_y, v_z) \\
&= u_x v_x + u_y v_y + u_z v_z \\
&= (v_x, v_y, v_z) \cdot (u_x, u_y, u_z) \\
&= \mathbf{v} \cdot \mathbf{u}
\end{aligned}$$

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (u_x, u_y, u_z) \cdot (v_x + w_x, v_y + w_y, v_z + w_z) \\
&= u_x(v_x + w_x) + u_y(v_y + w_y) + u_z(v_z + w_z) \\
&= u_x v_x + u_x w_x + u_y v_y + u_y w_y + u_z v_z + u_z w_z \\
&= (u_x v_x + u_y v_y + u_z) + (u_x w_x + u_y w_y + u_z w_z) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\end{aligned}$$

$$\begin{aligned}
k(\mathbf{u} \cdot \mathbf{v}) &= k(u_x v_x + u_y v_y + u_z v_z) \\
&= (ku_x)v_x + (ku_y)v_y + (ku_z)v_z \\
&= (k\mathbf{u}) \cdot \mathbf{v} \\
&= u_x(kv_x) + u_y(kv_y) + u_z(kv_z) \\
&= \mathbf{u} \cdot (k\mathbf{v})
\end{aligned}$$

$$\mathbf{v} \cdot \mathbf{v} = v_x v_x + v_y v_y + v_z v_z$$

$$= \sqrt{v_x^2 + v_y^2 + v_z^2}^{\,2}$$
$$= \|\mathbf{v}\|^2$$

$$\mathbf{0} \cdot \mathbf{v} = 0v_x + 0v_y + 0v_z = 0$$

11. Let $\mathbf{n} = (-2, 1)$. Decompose the vector $\mathbf{g} = (0, -9.8)$ into the sum of two orthogonal vectors, one parallel to $\mathbf{n}$ and the other orthogonal to $\mathbf{n}$. Also, draw the vectors relative to a 2D coordinate system.

Solution:

$$\mathbf{g}_{\parallel} = \text{proj}_{\mathbf{n}}(\mathbf{g}) = \frac{(\mathbf{g} \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-9.8}{5}(-2,1) = -1.96(-2,1) = (3.92, -1.96)$$

$$\mathbf{g}_{\perp} = \mathbf{g} - \mathbf{g}_{\parallel} = (0, -9.8) - (3.92, -1.96) = (-3.92, -7.84)$$

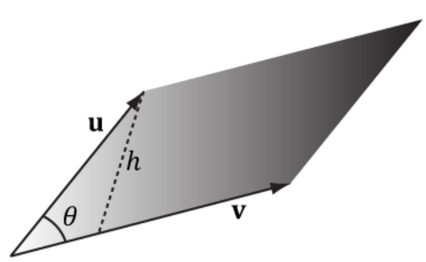
13. Let the following points define a triangle relative to some coordinate system: $\mathbf{A} = (0, 0, 0)$, $\mathbf{B} = (0, 1, 3)$, and $\mathbf{C} = (5, 1, 0)$. Find a vector orthogonal to this triangle. *Hint*: Find two vectors on two of the triangle's edges and use the cross product.

Solution:

$$\mathbf{u} = \mathbf{B} - \mathbf{A} = (0, 1, 3)$$
$$\mathbf{v} = \mathbf{C} - \mathbf{A} = (5,1,0)$$

$$\begin{aligned}\mathbf{n} = \mathbf{u} \times \mathbf{v} &= \left(u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x\right) \\ &= (0 - 3, 15 - 0, 0 - 5) \\ &= (-3, 15, -5)\end{aligned}$$

15. Prove that $\|\mathbf{u} \times \mathbf{v}\|$ gives the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$; see Figure below.



[Figure D.1: Parallelogram spanned by two 3D vectors $\mathbf{u}$ and $\mathbf{v}$; the parallelogram has base $\|\mathbf{v}\|$ and height h.]

Solution:

The area is the base times the height:

$$A = \|\mathbf{v}\| h$$

Using trigonometry, the height is given by $h = \|\mathbf{u}\|\sin(\theta)$. This, along with the application of Exercise 14, we can conclude:

$$A = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta) = \|\mathbf{u} \times \mathbf{v}\|$$

17. Prove that the cross product of two nonzero parallel vectors results in the null vector; that is, $\mathbf{u} \times k\mathbf{u} = 0$. *Hint*: Just use the cross product definition.

Solution:

$$\begin{aligned}\mathbf{u} \times k\mathbf{u} &= \left(u_y k u_z - u_z k u_y, u_z k u_x - u_x k u_z, u_x k u_y - u_y k u_x\right)\\ &= \left(k u_y u_z - k u_z u_y, k u_z u_x - k u_x u_z, k u_x u_y - k u_y u_x\right)\\ &= \mathbf{0}\end{aligned}$$

Chapter 2

1. Solve the following matrix equation for $\mathbf{X}$: $3\left(\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} - 2\mathbf{X}\right) = 2\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$.

Solution:

$$\begin{aligned} 3\left(\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} - 2\mathbf{X}\right) &= 2\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\\ \begin{bmatrix} -6 & 0 \\ 3 & 9 \end{bmatrix} - 6\mathbf{X} &= \begin{bmatrix} -4 & 0 \\ 2 & 6 \end{bmatrix}\\ -6\mathbf{X} &= \begin{bmatrix} -4 & 0 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ 3 & 9 \end{bmatrix}\\ -6\mathbf{X} &= \begin{bmatrix} 2 & 0 \\ -1 & -3 \end{bmatrix}\\ \mathbf{X} &= \begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix}\end{aligned}$$

3. Compute the transpose of the following matrices:

a) $[1, \quad 2, \quad 3]$, b) $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}$

Solution:

$$[1, \quad 2, \quad 3]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}^T = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

5. Show that

$$\mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} \leftarrow \mathbf{A}_{1,*}\mathbf{B} \rightarrow \\ \leftarrow \mathbf{A}_{2,*}\mathbf{B} \rightarrow \\ \leftarrow \mathbf{A}_{3,*}\mathbf{B} \rightarrow \end{bmatrix}$$

Solution:

Let i be an arbitrary row in $\mathbf{AB}$. By definition of matrix multiplication, we know that the ith row is given by

$$(\mathbf{AB})_{i,*} = [\mathbf{A}_{i,*} \cdot \mathbf{B}_{*,1} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,2} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,3}]$$

However, by definition of matrix multiplication, this is equal to the vector-matrix product $\mathbf{A}_{i,*}\mathbf{B}$. That is,

$$(\mathbf{AB})_{i,*} = [\mathbf{A}_{i,*} \cdot \mathbf{B}_{*,1} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,2} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,3}] = \mathbf{A}_{i,*}\mathbf{B}$$

Since i was an arbitrary row, we just substitute $i = 1,2,3$ to complete the proof:

$$\mathbf{AB} = \begin{bmatrix} (\mathbf{AB})_{1,*} \\ (\mathbf{AB})_{2,*} \\ (\mathbf{AB})_{3,*} \end{bmatrix} = \begin{bmatrix} \leftarrow \mathbf{A}_{1,*}\mathbf{B} \rightarrow \\ \leftarrow \mathbf{A}_{2,*}\mathbf{B} \rightarrow \\ \leftarrow \mathbf{A}_{3,*}\mathbf{B} \rightarrow \end{bmatrix}$$

7. Prove that the cross product can be expressed by the matrix product:

$$\mathbf{u} \times \mathbf{v} = [v_x \quad v_y \quad v_z] \begin{bmatrix} 0 & u_z & -u_y \\ -u_z & 0 & u_x \\ u_y & -u_x & 0 \end{bmatrix}$$

Solution:

$$[v_x \quad v_y \quad v_z]\begin{bmatrix} 0 & u_z & -u_y \\ -u_z & 0 & u_x \\ u_y & -u_x & 0 \end{bmatrix} = [u_y v_z - u_z v_y \quad u_z v_x - u_x v_z \quad u_x v_y - u_y v_x]$$
$$= \mathbf{u} \times \mathbf{v}$$

9. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Is $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 3/2 & 1/2 \end{bmatrix}$ the inverse of $\mathbf{A}$?

Solution:

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

Since $\mathbf{AB} \neq \mathbf{I}$ we can conclude that $\mathbf{B}$ is not the inverse of $\mathbf{A}$.

11. Find the inverse of the following matrices:

$$\begin{bmatrix} 21 & -4 \\ 10 & 7 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution:

For the 2×2 matrix, we have the formula:

$$\mathbf{A}^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}}\begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$
$$= \frac{1}{187}\begin{bmatrix} 7 & 4 \\ -10 & 21 \end{bmatrix}$$

To verify:

$$= \frac{1}{187}\begin{bmatrix} 7 & 4 \\ -10 & 21 \end{bmatrix}\begin{bmatrix} 21 & -4 \\ 10 & 7 \end{bmatrix}$$
$$= \frac{1}{187}\begin{bmatrix} 7(21) + 4(10) & 7(-4) + 4(7) \\ -10(21) + 10(21) & -10(-4) + 21(7) \end{bmatrix}$$
$$= \frac{1}{187}\begin{bmatrix} 187 & 0 \\ 0 & 187 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the 3×3 matrix, we use the formula:

$$\mathbf{A}^{-1} = \frac{\mathbf{A}^*}{\det \mathbf{A}}$$

The cofactor matrix is:

$$\mathbf{C_A} = \begin{bmatrix} (-1)^{1+1}\det\bar{\mathbf{A}}_{11} & (-1)^{1+2}\det\bar{\mathbf{A}}_{12} & (-1)^{1+3}\det\bar{\mathbf{A}}_{13} \\ (-1)^{2+1}\det\bar{\mathbf{A}}_{21} & (-1)^{2+2}\det\bar{\mathbf{A}}_{22} & (-1)^{2+3}\det\bar{\mathbf{A}}_{23} \\ (-1)^{3+1}\det\bar{\mathbf{A}}_{31} & (-1)^{3+2}\det\bar{\mathbf{A}}_{32} & (-1)^{3+3}\det\bar{\mathbf{A}}_{33} \end{bmatrix} = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

So $\mathbf{A}^* = \mathbf{C}_\mathbf{A}^T = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ and

$$\mathbf{A}^{-1} = \frac{1}{42}\begin{bmatrix} 21 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix}$$

To verify:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

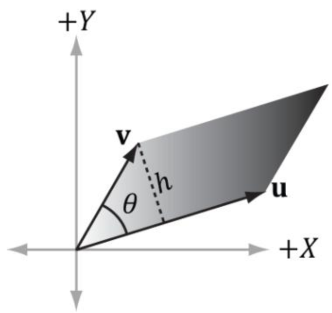
13. Show that $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$, assuming $\mathbf{A}$ is invertible.

Solution:

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}^T = \mathbf{I}$$
$$(\mathbf{A}^{-1})^T\mathbf{A}^T = (\mathbf{A}\mathbf{A}^{-1})^T = \mathbf{I}^T = \mathbf{I}$$

Therefore, $(\mathbf{A}^{-1})^T$ is the inverse of $\mathbf{A}^T$.

15. Prove that the 2D determinant $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ gives the signed area of the parallelogram spanned by $\mathbf{u} = (u_x, u_y)$ and $\mathbf{v} = (v_x, v_y)$. The result is positive if $\mathbf{u}$ can be rotated counterclockwise to coincide with $\mathbf{v}$ by an angle $\theta \in (0, \pi)$, and negative otherwise.



[Figure D.2: Parallelogram spanned by two 2D vectors **u** and **v**; the parallelogram has base $\|\mathbf{u}\|$ and height h.]

Solution:

The area is given by base times height:

$$\begin{aligned} A &= \|\mathbf{u}\|h \\ &= \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta \\ &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\cos^2\theta} \end{aligned}$$

Squaring both sides:

$$\begin{aligned} A^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta)^2 \\ &= (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2 \\ &= \left(u_x^2+u_y^2\right)\left(v_x^2+v_y^2\right) - \left(u_xv_x+u_yv_y\right)^2 \\ &= u_x^2v_x^2 + u_x^2v_y^2 + u_y^2v_x^2 + u_y^2v_y^2 - u_x^2v_x^2 - 2u_xv_xu_yv_y - u_y^2v_y^2 \\ &= u_x^2v_y^2 + u_y^2v_x^2 - 2u_xv_xu_yv_y \\ &= \left(u_xv_y - u_yv_x\right)^2 \end{aligned}$$

Taking the square root:

$$\begin{aligned} A &= \left|u_xv_y - u_yv_x\right| \\ &= \left|\det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right| \end{aligned}$$

17. Let $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$. Show that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$. This shows that matrix multiplication is associative for 2×2 matrices. (In fact, matrix multiplication is associative for general sized matrices, whenever the multiplication is defined.)

Solution:

For 2×2 matrices, we will just do the computations:

$$\mathbf{BC} = \begin{bmatrix} B_{11}C_{11} + B_{12}C_{21} & B_{11}C_{12} + B_{12}C_{22} \\ B_{21}C_{11} + B_{22}C_{21} & B_{21}C_{12} + B_{22}C_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} A_{11}(B_{11}C_{11} + B_{12}C_{21}) + A_{12}(B_{21}C_{11} + B_{22}C_{21}) & A_{11}(B_{11}C_{12} + B_{12}C_{22}) + A_{12}(B_{21}C_{12} + B_{22}C_{22}) \\ A_{21}(B_{11}C_{11} + B_{12}C_{21}) + A_{22}(B_{21}C_{11} + B_{22}C_{21}) & A_{21}(B_{11}C_{12} + B_{12}C_{22}) + A_{22}(B_{21}C_{12} + B_{22}C_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11}C_{11} + A_{11}B_{12}C_{21} + A_{12}B_{21}C_{11} + A_{12}B_{22}C_{21} & A_{11}B_{11}C_{12} + A_{11}B_{12}C_{22} + A_{12}B_{21}C_{12} + A_{12}B_{22}C_{22} \\ A_{21}B_{11}C_{11} + A_{21}B_{12}C_{21} + A_{22}B_{21}C_{11} + A_{22}B_{22}C_{21} & A_{21}B_{11}C_{12} + A_{21}B_{12}C_{22} + A_{22}B_{21}C_{12} + A_{22}B_{22}C_{22} \end{bmatrix}$$

$$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21})C_{11} + (A_{11}B_{12} + A_{12}B_{22})C_{21} & (A_{11}B_{11} + A_{12}B_{21})C_{12} + (A_{11}B_{12} + A_{12}B_{22})C_{22} \\ (A_{21}B_{11} + A_{22}B_{21})C_{11} + (A_{21}B_{12} + A_{22}B_{22})C_{21} & (A_{21}B_{11} + A_{22}B_{21})C_{12} + (A_{21}B_{12} + A_{22}B_{22})C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11}C_{11} + A_{12}B_{21}C_{11} + A_{11}B_{12}C_{21} + A_{12}B_{22}C_{21} & A_{11}B_{11}C_{12} + A_{12}B_{21}C_{12} + A_{11}B_{12}C_{22} + A_{12}B_{22}C_{22} \\ A_{21}B_{11}C_{11} + A_{22}B_{21}C_{11} + A_{21}B_{12}C_{21} + A_{22}B_{22}C_{21} & A_{21}B_{11}C_{12} + A_{22}B_{21}C_{12} + A_{21}B_{12}C_{22} + A_{22}B_{22}C_{22} \end{bmatrix}$$

Comparing the terms element-by-element, we see $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

Chapter 3

1. Let $\tau: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\tau(x, y, z) = (x + y, x - 3, z)$. Is τ a linear transformation? If it is, find its standard matrix representation.

Solution:

If τ is linear, then we must have:

1. $\tau(\mathbf{u} + \mathbf{v}) = \tau(\mathbf{u}) + \tau(\mathbf{v})$
2. $\tau(k\mathbf{u}) = k\tau(\mathbf{u})$

Let $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{v} = (v_x, v_y, v_z)$.

$$\begin{aligned}
\tau(\mathbf{u} + \mathbf{v}) &= \tau(u_x + v_x, u_y + v_y, u_z + v_z) \\
&= (u_x + v_x + u_y + v_y, u_x + v_x - 3, u_z + v_z) \\
&= (u_x + u_y, u_x - 3, u_z) + (v_x + v_y, v_x, v_z) \\
&= \tau(\mathbf{u}) + (v_x + v_y, v_x, v_z) \\
&= \tau(\mathbf{u}) + (v_x + v_y, v_x - 3 + 3, v_z) \\
&= \tau(\mathbf{u}) + (v_x + v_y, v_x - 3, v_z) + (0, 3, 0) \\
&= \tau(\mathbf{u}) + \tau(\mathbf{v}) + (0, 3, 0)
\end{aligned}$$

So $\tau(\mathbf{u} + \mathbf{v}) \neq \tau(\mathbf{u}) + \tau(\mathbf{v})$; therefore τ is not a linear transformation.

3. Assume that $\tau: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation. Further suppose that $\tau(1, 0, 0) = (3, 1, 2)$, $\tau(0, 1, 0) = (2, -1, 3)$, and $\tau(0, 0, 1) = (4, 0, 2)$. Find $\tau(1, 1, 1)$.

Solution:

We are given that τ is a linear transformation and its behavior on the standard basis vectors $\tau(\mathbf{i}), \tau(\mathbf{j}), \tau(\mathbf{k})$. Therefore, the standard matrix representation of τ is:

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$$

Then

$$\tau(1,1,1) = [1,1,1]\begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 3 \\ 4 & 0 & 2 \end{bmatrix} = [3+2+4, 1-1, 2+3+2] = [9,0,7]$$

5. Build a rotation matrix that rotates 30° along the axis $(1,1,1)$.

Solution:

$$(x,y,z) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$
$$c = \cos 30° = \frac{\sqrt{3}}{2}$$
$$s = \sin 30° = \frac{1}{2}$$

$$\mathbf{R_n} = \begin{bmatrix} c+(1-c)x^2 & (1-c)xy+sz & (1-c)xz-sy \\ (1-c)xy-sz & c+(1-c)y^2 & (1-c)yz+sx \\ (1-c)xz+sy & (1-c)yz-sx & c+(1-c)z^2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\sqrt{3}}{2}+\left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3} & \left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3}+\frac{1}{2}\frac{1}{\sqrt{3}} & \left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3}-\frac{1}{2}\frac{1}{\sqrt{3}} \\ \left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3}-\frac{1}{2}\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2}+\left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3} & \left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3}+\frac{1}{2}\frac{1}{\sqrt{3}} \\ \left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3}+\frac{1}{2}\frac{1}{\sqrt{3}} & \left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3}-\frac{1}{2}\frac{1}{\sqrt{3}} & \frac{\sqrt{3}}{2}+\left(1-\frac{\sqrt{3}}{2}\right)\frac{1}{3} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\sqrt{3}}{2}+\frac{2-\sqrt{3}}{6} & \left(\frac{1}{3}-\frac{\sqrt{3}}{6}\right)+\frac{\sqrt{3}}{6} & \left(\frac{1}{3}-\frac{\sqrt{3}}{6}\right)-\frac{\sqrt{3}}{6} \\ \left(\frac{1}{3}-\frac{\sqrt{3}}{6}\right)-\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{2}+\frac{2-\sqrt{3}}{6} & \left(\frac{1}{3}-\frac{\sqrt{3}}{6}\right)+\frac{\sqrt{3}}{6} \\ \left(\frac{1}{3}-\frac{\sqrt{3}}{6}\right)+\frac{\sqrt{3}}{6} & \left(\frac{1}{3}-\frac{\sqrt{3}}{6}\right)-\frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{2}+\frac{2-\sqrt{3}}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2\sqrt{3}+2}{6} & \frac{1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{2\sqrt{3}+2}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{2\sqrt{3}+2}{6} \end{bmatrix}$$

7. Build a single transformation matrix that first scales 2 units on the *x*-axis, -3 units on the *y*-axis, and keeps the *z*-dimension unchanged, and then translates 4 units on the *x*-axis, no units on the *y*-axis, and -9 units on the *z*-axis.

Solution:

$$\mathbf{S} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & -9 & 1 \end{bmatrix}$$

Then the desired transformation matrix is obtained by the product

$$\mathbf{M} = \mathbf{ST} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & -9 & 1 \end{bmatrix}$$

9. Redo Example 3.2, but this time scale the square 1.5 units on the *x*-axis, 0.75 units on the *y*-axis, and leave the *z*-axis unchanged. Graph the geometry before and after the transformation to confirm your work.

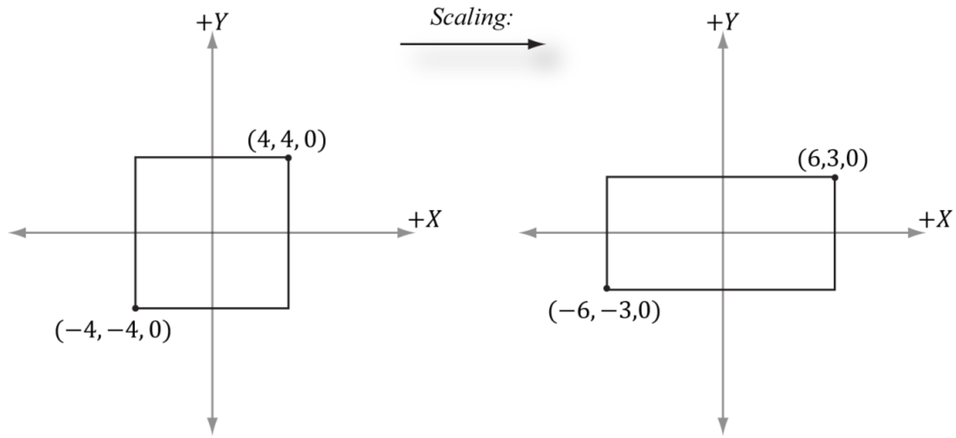
Solution:

The corresponding scaling matrix is:

$$\mathbf{S} = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now to actually scale (transform) the square, we multiply both the minimum point and maximum point by this matrix:

$$[-4, -4, 0] \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [-6, -3, 0] \qquad [4, 4, 0] \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & .75 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [6, 3, 0]$$



[Figure D.3: Scaling transform.]

11. Redo Example 3.4, but this time translate the square -5 units on the *x*-axis, -3.0 units on the *y*-axis, and 4.0 units on the *z*-axis. Graph the geometry before and after the transformation to confirm your work.

Solution:

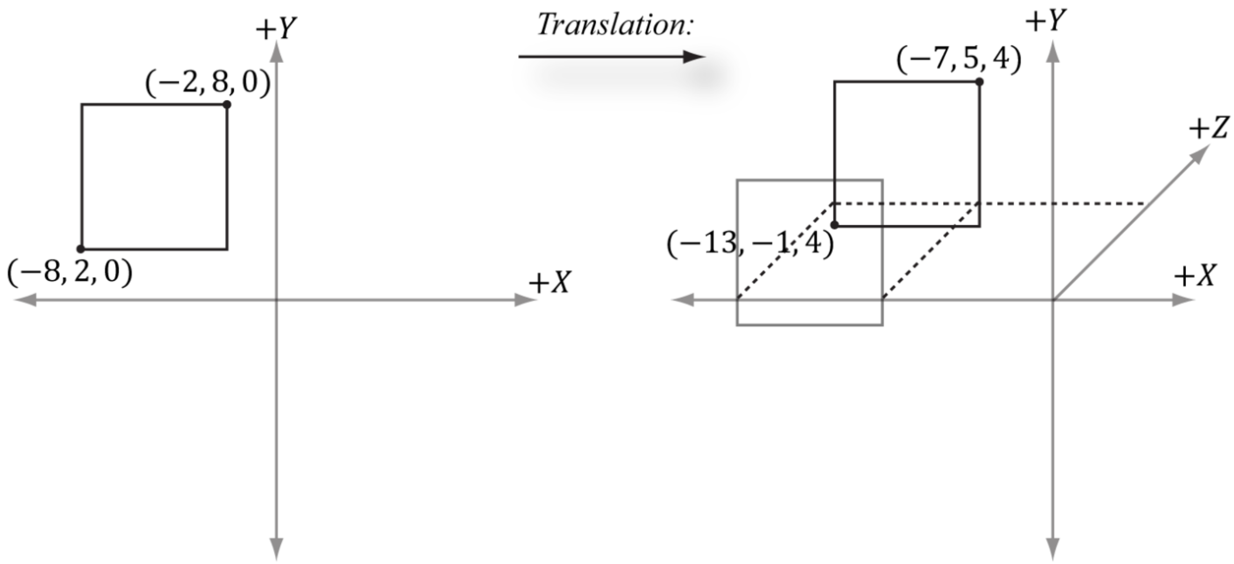
The corresponding translation matrix is:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & -3 & 4 & 1 \end{bmatrix}$$

Now to actually translate (transform) the square, we multiply both the minimum point and maximum point by this matrix:

$$[-8, \quad 2, \quad 0, \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & -3 & 4 & 1 \end{bmatrix} = [-13, \quad -1, \quad 4, \quad 1]$$

$$[-2, \quad 8, \quad 0, \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & -3 & 4 & 1 \end{bmatrix} = [-7, \quad 5, \quad 4, \quad 1]$$



[Figure D.4: Translation transform.]

13. Prove that the rows of $\mathbf{R_y}$ are orthonormal. For a more computational intensive exercise, the reader can do this for the general rotation matrix (rotation matrix about an arbitrary axis), too.

$$\mathbf{R_y} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

Solution:

Let $\mathbf{r}_1 = (\cos\theta, 0, -\sin\theta)$, $\mathbf{r}_2 = (0, 1, 0)$, and $\mathbf{r}_3 = (\sin\theta, 0, \cos\theta)$.

First we show all the rows are unit length:

$$\|\mathbf{r}_1\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$
$$\|\mathbf{r}_2\| = 1$$
$$\|\mathbf{r}_3\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

Next we show the rows are mutually orthogonal:

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0$$
$$\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$$
$$\mathbf{r}_1 \cdot \mathbf{r}_3 = \cos\theta\sin\theta - \sin\theta\cos\theta = 0$$

14. Prove the matrix $\mathbf{M}$ is orthogonal if and only if $\mathbf{M}^T = \mathbf{M}^{-1}$.

Solution:

For concreteness, we will prove for 3×3 matrices since 3D rotation matrices are the only kind of orthogonal matrices we care about in this book. However, the same argument we present generalizes to $n \times n$ matrices.

Suppose $\mathbf{M}$ is a 3×3 orthogonal matrix:

$$\mathbf{M} = \begin{bmatrix} \leftarrow \mathbf{r}_1 \rightarrow \\ \leftarrow \mathbf{r}_2 \rightarrow \\ \leftarrow \mathbf{r}_3 \rightarrow \end{bmatrix} \quad \text{and} \quad \mathbf{M}^T = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

Then, by the definition of matrix multiplication $(\mathbf{MM}^T)_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$. But, because $\mathbf{M}$ is orthogonal we have that

$$(\mathbf{MM}^T)_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

It is true that $i = j$ only for the diagonal elements of the matrix. Thus the resulting matrix has 1's along the diagonal and zeros everywhere else. But this is exactly the identity matrix. Hence, $\mathbf{MM}^T = \mathbf{I}$. A similar argument shows $\mathbf{M}^T\mathbf{M} = \mathbf{I}$. Therefore, we must have that $\mathbf{M}^T = \mathbf{M}^{-1}$.

Now suppose that $\mathbf{M}^T = \mathbf{M}^{-1}$. In particular, this implies that $\mathbf{MM}^T = \mathbf{I}$**.** In turn, this means:

$$(\mathbf{MM}^T)_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Therefore, the row vectors or $\mathbf{M}$ are mutually orthogonal and unit length; thus, $\mathbf{M}$ is orthogonal.

15. Compute:

$$[x, y, z, 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix} \quad \text{and} \quad [x, y, z, 0]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix}$$

Does the translation translate points? Does the translation translate vectors? Why does it not make sense to translate the coordinates of a vector in standard position?

Solution:

$$[x, y, z, 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix} = [x + b_x, y + b_y, z + b_z, 1]$$

The translation translates points.

$$[x, y, z, 0]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix} = [x, y, z, 0]$$

The translation does not translate vectors. Translation does not make sense for vectors because a vector only describes direction and magnitude, independent of location.

17. Suppose that we have frames *A* and *B*. Let $\mathbf{p}_A = (1, -2, 0)$ and $\mathbf{q}_A = (1, 2, 0)$ represent a point and force, respectively, relative to frame *A*. Moreover, let $\mathbf{Q}_B = (-6, 2, 0)$, $\mathbf{u}_B = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\mathbf{v}_B = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, and $\mathbf{w}_B = (0, 0, 1)$ describe frame *A* with coordinates relative to frame *B*. Build the change of coordinate matrix that maps frame *A* coordinates into frame *B* coordinates, and find $\mathbf{p}_B = (x, y, z)$ and $\mathbf{q}_B = (x, y, z)$. Draw a picture on graph paper to verify that your answer is reasonable.

Solution:

From Equation 3.9, the change of coordinate matrix is:

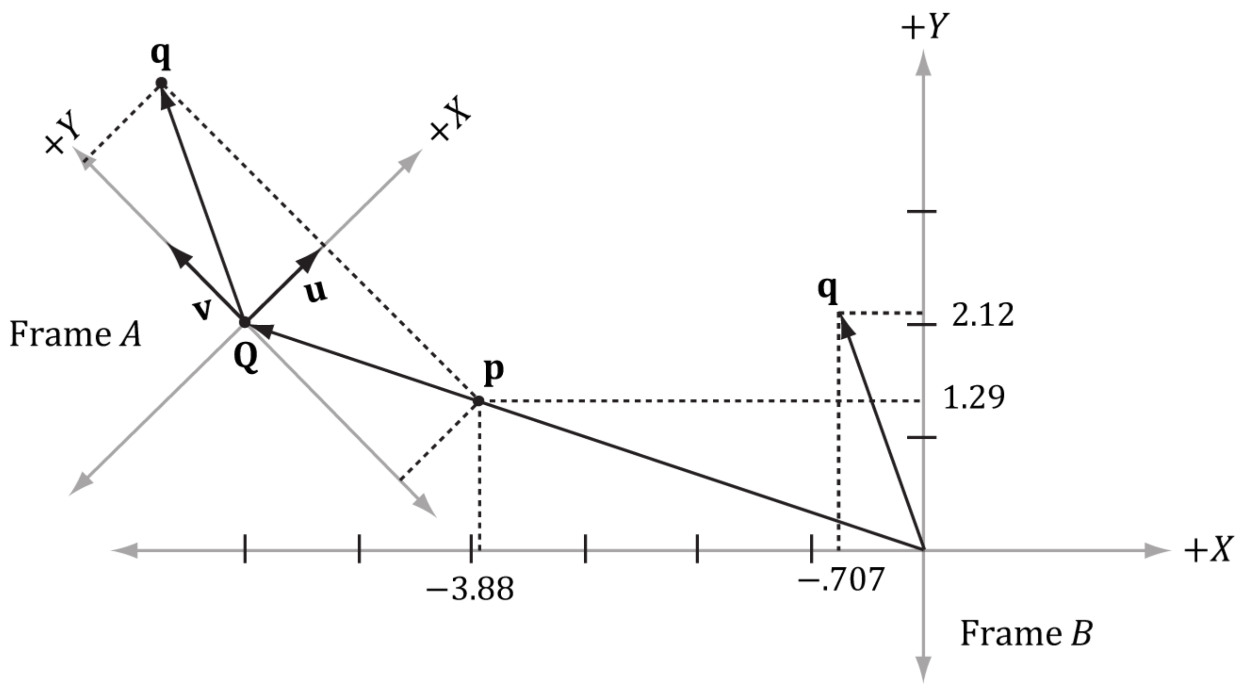
$$\begin{bmatrix} \leftarrow \mathbf{u}_B \rightarrow \\ \leftarrow \mathbf{v}_B \rightarrow \\ \leftarrow \mathbf{w}_B \rightarrow \\ \leftarrow \mathbf{Q}_B \rightarrow \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{bmatrix}$$

Then to transform points and vectors from frame *A* into frame *B* we multiply the frame *A* coordinate vectors by the matrix:

$$\begin{aligned} \mathbf{p}_B &= [1 \quad -2 \quad 0 \quad 1]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{bmatrix} \\ &= \left[\frac{3}{\sqrt{2}} - 6 \quad -\frac{1}{\sqrt{2}} + 2 \quad 0 \quad 1\right] \\ &\approx [-3.88 \quad 1.29 \quad 0 \quad 1] \end{aligned}$$

$$\mathbf{q}_B = [1 \quad 2 \quad 0 \quad 0]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$
$$\approx [-.707 \quad 2.12 \quad 0 \quad 0]$$



[Figure D.5: Change of coordinates from frame *A* to frame *B*.]

19. Consider the triangle defined by the points $\mathbf{p}_1 = (0, 0, 0)$, $\mathbf{p}_2 = (0,1,0)$, and $\mathbf{p}_3 = (2, 0, 0)$. Graph the following points:

a) $\frac{1}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2 + \frac{1}{3}\mathbf{p}_3$
b) $0.7\mathbf{p}_1 + 0.2\mathbf{p}_2 + 0.1\mathbf{p}_3$
c) $0.0\mathbf{p}_1 + 0.5\mathbf{p}_2 + 0.5\mathbf{p}_3$
d) $-0.2\mathbf{p}_1 + 0.6\mathbf{p}_2 + 0.6\mathbf{p}_3$
e) $0.6\mathbf{p}_1 + 0.5\mathbf{p}_2 - 0.1\mathbf{p}_3$
f) $0.8\mathbf{p}_1 - 0.3\mathbf{p}_2 + 0.5\mathbf{p}_3$

What is special about the point in part (a)? What would be the barycentric coordinates of $\mathbf{p}_2$ and the point $(1, 0, 0)$ in terms of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$? Can you make a conjecturer about where the point $\mathbf{p}$ will be located relative to the triangle if one of the barycentric coordinates is negative?

Solution:

a) $\frac{1}{3}\mathbf{p}_1 + \frac{1}{3}\mathbf{p}_2 + \frac{1}{3}\mathbf{p}_3 = \frac{1}{3}(0, 0, 0) + \frac{1}{3}(0,1,0) + \frac{1}{3}(2, 0, 0) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$

b) $0.7\mathbf{p}_1 + 0.2\mathbf{p}_2 + 0.1\mathbf{p}_3 = 0.7(0, 0, 0) + 0.2(0,1,0) + 0.1\mathbf{p}_3(2, 0, 0) = (0.2, 0.2, 0)$

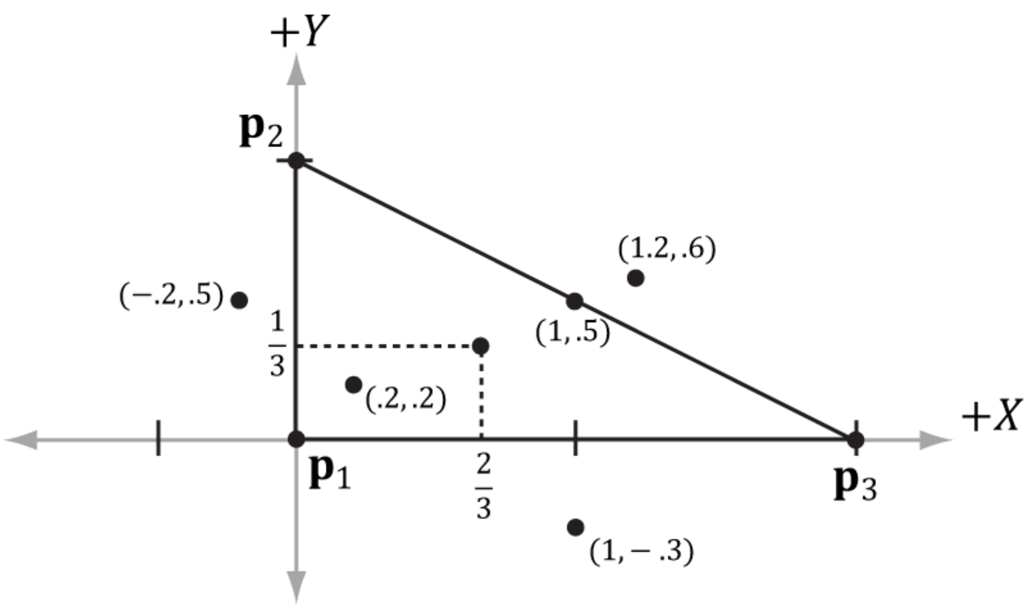
c) $0.0\mathbf{p}_1 + 0.5\mathbf{p}_2 + 0.5\mathbf{p}_3 = 0.0(0,0,0) + 0.5(0,1,0) + 0.5\mathbf{p}_3(2,0,0) = (1, 0.5, 0)$

d) $-0.2\mathbf{p}_1 + 0.6\mathbf{p}_2 + 0.6\mathbf{p}_3 = -0.2(0,0,0) + 0.6(0,1,0) + 0.6\mathbf{p}_3(2,0,0) = (1.2, 0.6, 0)$

e) $0.6\mathbf{p}_1 + 0.5\mathbf{p}_2 - 0.1\mathbf{p}_3 = 0.6(0,0,0) + 0.5(0,1,0) - 0.1\mathbf{p}_3(2,0,0) = (-0.2, 0.5, 0)$

f) $0.8\mathbf{p}_1 - 0.3\mathbf{p}_2 + 0.5\mathbf{p}_3 = 0.8(0,0,0) - 0.3(0,1,0) + 0.5\mathbf{p}_3(2,0,0) = (1, -0.3, 0)$

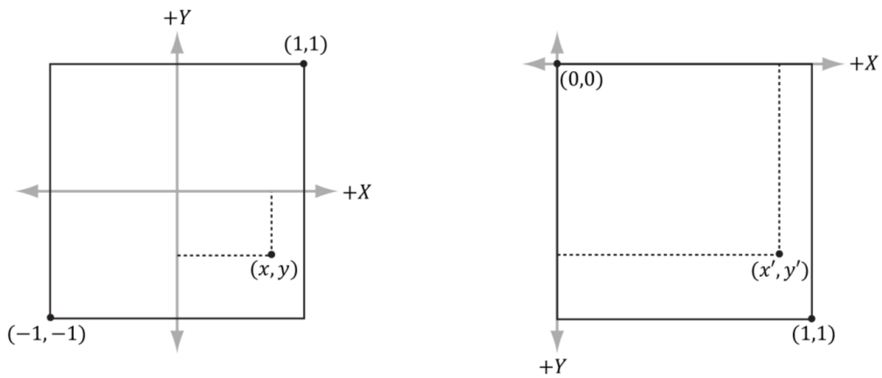
The point in part (a) is the centroid. $\mathbf{p}_2 = 0\mathbf{p}_1 + 1\mathbf{p}_2 + 0\mathbf{p}_3$ so the barycentric coordinates are $(0,1,0)$. $(1,0,0) = \frac{1}{2}\mathbf{p}_1 + \frac{1}{2}\mathbf{p}_3$ so the barycentric coordinates are $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$. For negative barycentric coordinates, the points lie outside the triangle.



[Figure D.6: Plotting barycentric coordinates.]

21. Consider Figure 3.16. A common change of coordinate transformation in computer graphics is to map coordinates from frame *A* (the square $[-1,1]^2$) to frame *B* (the square $[0,1]^2$ where the *y*-axes aims opposite to the one in Frame *A*). Prove that the change of coordinate transformation from Frame *A* to Frame *B* is given by:

$$[x, \quad y, \quad 0 \quad 1]\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0 & 1 \end{bmatrix} = [x', \quad y', \quad 0 \quad 1]$$



[Figure D.7: Change of coordinates from frame *A* (the square $[-1,1]^2$) to frame *B* (the square $[0,1]^2$ where the *y*-axes aims opposite to the one in Frame *A*)]

Solution:

From Equation 3.9, the change of coordinate matrix is:

$$[x', \quad y', \quad z', \quad w] = [x, \quad y, \quad z, \quad w]\begin{bmatrix} \leftarrow \mathbf{u}_B \rightarrow \\ \leftarrow \mathbf{v}_B \rightarrow \\ \leftarrow \mathbf{w}_B \rightarrow \\ \leftarrow \mathbf{Q}_B \rightarrow \end{bmatrix}$$

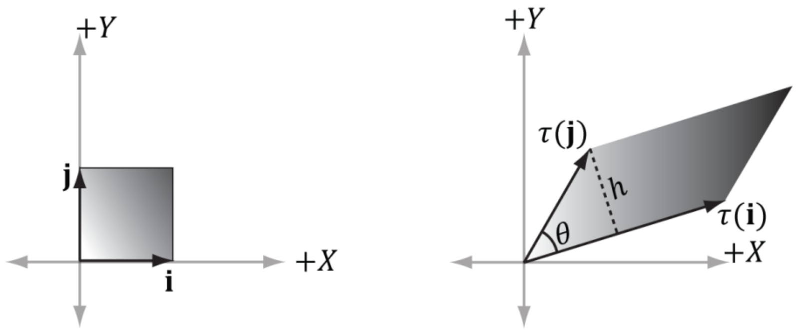
That is, we need to describe the coordinate system of frame *A* (origin and axes) with coordinates relative to frame *B*. (We use homogeneous coordinates.) The origin in frame *A* is the center of the square, which has coordinates $\mathbf{Q}_B = (0.5, 0.5, 0, 1)$. The *x*-axis of frame *A* aims in the same direction as frame *B*, and, along this axis, one unit in frame *A* is half a unit in frame *B;* therefore, $\mathbf{u}_B = (0.5, 0, 0, 0)$. The *y*-axis of frame *A* aims in the opposite direction as frame *B*, and along this axis, one unit in frame *A* is half a unit in frame *B*; therefore, $\mathbf{v}_B = (0, -0.5, 0, 0)$. Since this is a 2D problem, we will just assume that the *z*-axes are the same in both coordinate systems; therefore, $\mathbf{w}_B = (0, 0, 1, 0)$. Substituting these numbers into the above yields the desired transformation matrix:

$$\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

23. Consider the transformation τ that warps a square into a parallelogram given by:

$$\tau(x, y) = (3x + y, x + 2y)$$

Find the standard matrix representation of this transformation, and show that the determinant of the transformation matrix is equal to the area of the parallelogram spanned by $\tau(\mathbf{i})$ and $\tau(\mathbf{j})$.



[FigD.8.bmp: Transformation that maps square into parallelogram.]

Solution:

$$\tau(1,0) = (3,1)$$
$$\tau(0,1) = (1,2)$$

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det \mathbf{A} = 6 - 1 = 5$$

The area of the parallelogram spanned by by $\tau(\mathbf{i})$ and $\tau(\mathbf{j})$ is given by:

$$\begin{aligned} Area &= Base \times Height \\ &= \|\tau(\mathbf{i})\|\|\tau(\mathbf{j})\| \sin\theta \\ &= \|\tau(\mathbf{i}) \times \tau(\mathbf{j})\| \end{aligned}$$

The cross product is not defined in 2D, but we can do a trick and augment to 3D space with $z = 0$ so that we can use the cross product. Note that using $z = 0$ does not affect the magnitude or direction of the vectors.

$$(3,1,0) \times (1,2,0) = (0,0,5)$$

$$\begin{aligned} &= \|(3,1,0) \times (1,2,0)\| \\ &= \|(0,0,5)\| \\ &= 5 \end{aligned}$$

Also see Chapter 2, Exercise 15. So observe that the area of the parallelogram (square) defined by the standard basis vectors **i** and **j** is 1, and the area of the parallelogram spanned by $\tau(\mathbf{i})$ and $\tau(\mathbf{j})$ is 5. Therefore, the transformation τ changed the volume (or area in 2D) from 1 to 5 when it warped it from a unit square to the parallelogram spanned by $\tau(\mathbf{i})$ and $\tau(\mathbf{j})$.

25. A rotation matrix can be characterized algebraically as an orthogonal matrix with determinant equal to 1. If we reexamine Figure 3.7 along with Exercise 24 this makes sense; the rotated basis vectors $\tau(\mathbf{i})$, $\tau(\mathbf{j})$, and $\tau(\mathbf{k})$ are unit length and mutually orthogonal; moreover, rotation does not change the size of the object, so the determinant should be 1. Show that the

product of two rotation matrices $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}$ is a rotation matrix. That is, show $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$ (to show $\mathbf{R}$ is orthogonal), and show $\det \mathbf{R} = 1$.

Solution:

$$\mathbf{R}_1\mathbf{R}_2(\mathbf{R}_1\mathbf{R}_2)^T = \mathbf{R}_1\mathbf{R}_2\mathbf{R}_2^T\mathbf{R}_1^T = \mathbf{R}_1\mathbf{R}_1^T = \mathbf{I}$$
$$(\mathbf{R}_1\mathbf{R}_2)^T\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2^T\mathbf{R}_1^T\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2^T\mathbf{R}_2 = \mathbf{I}$$

Using the fact that $\det(\mathbf{AB}) = \det \mathbf{A} \cdot \det \mathbf{B}$:

$$\det(\mathbf{R}_1\mathbf{R}_2) = \det(\mathbf{R}_1)\det(\mathbf{R}_2) = 1$$

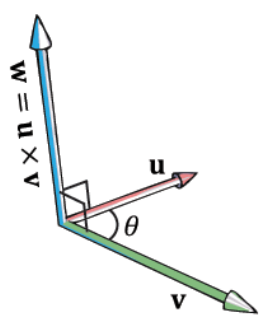
27. Find a scaling, rotation, and translation matrix whose product transforms the line segment with start point $\mathbf{p} = (0,0,0)$ and endpoint $\mathbf{q} = (0,0,1)$ into the line segment with length 2, parallel to the vector $(1,1,1)$, with start point $(3,1,2)$.

Solution:

The line segment currently aims along the *z*-axis with a length of 1. To make it have a length of 2, we first scale it 2-units on the *z*-axis. In order to rotate the line segment so that it is parallel to $(1,1,1)$ we need to rotate by an angle θ in the plane that contains the vectors $(0,0,1)$ and $(1,1,1)$ (see Figure). The axis of rotation is given by $\mathbf{u} \times \mathbf{v} = (0,0,1) \times (1,1,1) = (-1,1,0)$. The angle we need to rotate is given by the angle between the vectors $(0,0,1)$ and $(1,1,1)$:

$$\theta = \cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos^{-1}\frac{1}{\sqrt{3}} = 54.73°$$

Finally, we must apply a translation $\mathbf{T}(3,1,2)$.

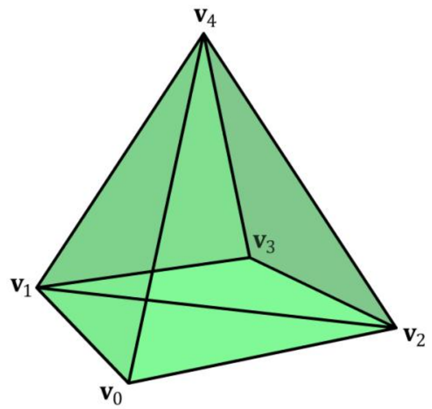


[FigD.9: To rotate **u** so that it aims in the same direction as **v**, we must rotate **u** by an angle θ about the axis $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.]

Chapter 5

1. Construct the vertex and index list of a pyramid, as shown in Figure 5.35.

Solution:



[FigD.10: Vertices of the pyramid.]

```
Vertex vertices[5] = {v0, v1, v2, v3, v4};
UINT indices[] = {
    0, 1, 2,
    1, 3, 2,
    1, 4, 0,
    0, 4, 2,
    2, 4, 3,
    3, 4, 1
};
```

3. Relative to the world coordinate system, suppose that the camera is positioned at $(-20, 35, -50)$ and looking at the point $(10, 0, 30)$. Compute the view matrix assuming (0,1,0) describes the "up" direction in the world.

Solution:

Figure 5.20 shows the setup. Let $\mathbf{Q} = (-20, 35, -50)$, $\mathbf{T} = (10, 0, 30)$ and $\mathbf{j} = (0, 1, 0)$. The direction the camera is looking is given by:

$$\mathbf{w} = \frac{\mathbf{T} - \mathbf{Q}}{\|\mathbf{T} - \mathbf{Q}\|} = \frac{(30, -35, 80)}{5\sqrt{341}} = (.3249, -.3791, .8664)$$

This vector describes the local *z*-axis of the camera. A unit vector that aims to the "right" of **w** is given by:

$$\mathbf{u} = \frac{\mathbf{j} \times \mathbf{w}}{\|\mathbf{j} \times \mathbf{w}\|} = \frac{(.8664, 0, -.3249)}{.9254} = (.9363, 0, -.3511)$$

This vector describes the local *x*-axis of the camera. Finally, a vector that describes the local *y*-axis of the camera is given by:

$$\mathbf{v} = \mathbf{w} \times \mathbf{u} = (.1331, .9253, .3549)$$

The view matrix is:

$$\mathbf{V} = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ -\mathbf{Q}\cdot\mathbf{u} & -\mathbf{Q}\cdot\mathbf{v} & -\mathbf{Q}\cdot\mathbf{w} & 1 \end{bmatrix} = \begin{bmatrix} .9363 & .1331 & .3249 & 0 \\ 0 & .9253 & -.3791 & 0 \\ -.3511 & .3549 & .8664 & 0 \\ 1.170 & -11.98 & 63.09 & 1 \end{bmatrix}$$

You can verify the answer by writing the following code and inspecting the matrix elements:

```
XMMATRIX K = XMMatrixLookAtLH(
            XMLoadFloat3(&XMFLOAT3(-20.0f, 35.0f, -50.0f)),
            XMLoadFloat3(&XMFLOAT3(10.0f, 0.0f, 30.0f)),
            XMLoadFloat3(&XMFLOAT3(0.0f, 1.0f, 0.0f)));
```

5. Suppose that the view window has height 4. Find the distance d from the origin the view window must be to create a vertical field of view angle $\theta = 60°$.

Solution:

$$\tan 30° = \frac{2}{d}$$
$$\therefore d = \frac{2}{\tan 30°} = 3.464$$

7. Suppose that you are given the following perspective projection matrix with fixed A, B, C, D:

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 1 \\ 0 & 0 & D & 0 \end{bmatrix}$$

Find the vertical field of view angle α the aspect ratio r, and the near and far plane values that were used to build this matrix in terms of A, B, C, D. That is, solve the following equations:

$$A = \frac{1}{r\tan(\alpha/2)}$$
$$B = \frac{1}{\tan(\alpha/2)}$$
$$C = \frac{f}{f-n}$$
$$D = \frac{-nf}{f-n}$$

Solving these equations will give you formulas for extracting the vertical field of view angle α the aspect ratio r, and the near and far plane values from any perspective projection matrix of the kind described in this book.

Solution:

Dividing the second equation by the first equation yields:

$$\frac{B}{A} = \frac{1}{\tan(\alpha/2)} \cdot \frac{r\tan(\alpha/2)}{1} = r$$

The second equation implies:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{1}{B}$$
$$\therefore \alpha = 2\tan^{-1}\left(\frac{1}{B}\right)$$

We solve the third equation for f:

$$C = \frac{f}{f-n}$$
$$Cf - f - Cn = 0$$
$$f = \frac{Cn}{C-1}$$

Now we plug f into the fourth equation and solve for n:

$$D = \frac{-nf}{f-n}$$
$$Df - Dn = -nf$$
$$D\frac{Cn}{C-1} - Dn = -n\frac{Cn}{C-1}$$
$$CDn - Dn(C-1) = -Cn^2$$
$$Dn = -Cn^2$$
$$D = -Cn$$
$$n = -\frac{D}{C}$$

Now,

$$f = \frac{Cn}{C-1} = \frac{-D}{C-1} = \frac{D}{1-C}$$

In summary,

$$r = \frac{B}{A}$$
$$\alpha = 2\tan^{-1}\left(\frac{1}{B}\right)$$
$$n = -\frac{D}{C}$$
$$f = \frac{D}{1 - C}$$

8. For projective texturing algorithms, we multiply an affine transformation matrix $\mathbf{T}$ after the projection matrix. Prove that it does not matter if we do the perspective divide before or after multiplying by $\mathbf{T}$. Let, $\mathbf{v}$ be a 4D vector, $\mathbf{P}$ be a projection matrix, $\mathbf{T}$ be a 4×4 affine transformation matrix, and let a w subscript denote the w-coordinate of a 4D vector, prove:

$$\left(\frac{\mathbf{vP}}{(\mathbf{vP})_w}\right)\mathbf{T} = \frac{(\mathbf{vPT})}{(\mathbf{vPT})_w}$$

Solution:

Using the properties of matrix algebra, we have:

$$\left(\frac{\mathbf{vP}}{(\mathbf{vP})_w}\right)\mathbf{T} = \frac{1}{(\mathbf{vP})_w}(\mathbf{vPT})$$

But $\mathbf{T}$ is given as an affine transformation matrix, which means it does not modify the w-coordinate. Hence, we have $(\mathbf{vP})_w = \big((\mathbf{vP})\mathbf{T}\big)_w = (\mathbf{vPT})_w$. Hence,

$$\left(\frac{\mathbf{vP}}{(\mathbf{vP})_w}\right)\mathbf{T} = \frac{1}{(\mathbf{vP})_w}(\mathbf{vPT}) = \frac{(\mathbf{vPT})}{(\mathbf{vPT})_w}$$

10. Let $[x, y, z, 1]$ be the coordinates of a point in view space, and let $[x_{ndc}, y_{ndc}, z_{ndc}, 1]$ be the coordinates of the same point in NDC space. Prove that you can transform from NDC space to view space in the following way:

$$[x_{ndc}, y_{ndc}, z_{ndc}, 1]\mathbf{P}^{-1} = \left[\frac{x}{z}, \frac{y}{z}, 1, \frac{1}{z}\right] \xrightarrow{divide\ \ by\ w} [x, y, z, 1]$$

Explain why you need the division by w. Would you need the division by w if you were transforming from homogeneous clip space to view space?

Solution:

Recall from Equation 5.1 and §5.6.3.5 that:

$$\begin{aligned} x_{ndc} &= \frac{x}{rz\tan(\alpha/2)} \\ y_{ndc} &= \frac{y}{z\tan(\alpha/2)} \\ z_{ndc} &= \frac{f}{f-n} + \frac{-nf}{(f-n)z} \end{aligned}$$

where (x, y, z) are view space coordinates.

$$[x_{ndc}, y_{ndc}, z_{ndc}, 1]\begin{bmatrix} r\tan\left(\frac{\alpha}{2}\right) & 0 & 0 & 0 \\ 0 & \tan\left(\frac{\alpha}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & -\frac{f-n}{nf} \\ 0 & 0 & 1 & \frac{1}{n} \end{bmatrix}$$

$$\begin{aligned} &= \left[x_{ndc}\, r\tan\left(\frac{\alpha}{2}\right), y_{ndc}\tan\left(\frac{\alpha}{2}\right), 1, -z_{ndc}\frac{f-n}{nf} + \frac{1}{n}\right] \\ &= \left[\frac{x}{rz\tan(\alpha/2)} r\tan\left(\frac{\alpha}{2}\right), \frac{y}{z\tan(\alpha/2)}\tan\left(\frac{\alpha}{2}\right), 1, -\left(\frac{f}{f-n} + \frac{-nf}{(f-n)z}\right)\frac{f-n}{nf} + \frac{1}{n}\right] \\ &= \left[\frac{x}{z}, \frac{y}{z}, 1, \left(\frac{-f}{f-n} + \frac{nf}{(f-n)z}\right)\frac{f-n}{nf} + \frac{1}{n}\right] \\ &= \left[\frac{x}{z}, \frac{y}{z}, 1, \left(\frac{-f}{nf} + \frac{nf}{znf}\right) + \frac{1}{n}\right] \\ &= \left[\frac{x}{z}, \frac{y}{z}, 1, \frac{-1}{n} + \frac{1}{z} + \frac{1}{n}\right] \\ &= \left[\frac{x}{z}, \frac{y}{z}, 1, \frac{1}{z}\right] \end{aligned}$$

$$\left[\frac{x}{z}, \frac{y}{z}, 1, \frac{1}{z}\right] \xrightarrow{divide\ \ by\ w} [x, y, z, 1]$$

The steps to go from view space to NDC space are multiplying by the projection matrix, followed by a division by $w = z$. Therefore, to transform back from NDC space to view space, we multiply by the inverse projection matrix followed by a division by $w = 1/z$ (which is equivalent to *multiplying* by $w = z$).

If you are given points in homogeneous clip space (before the divide by $w = z$), then you do not need to divide by w in the inverse transformation. In homogeneous clip space, the point $[x_{ndc}, y_{ndc}, z_{ndc}, 1]$ has coordinates $[zx_{ndc}, zy_{ndc}, zz_{ndc}, z]$:

$$[zx_{ndc}, zy_{ndc}, zz_{ndc}, z]\begin{bmatrix} r\tan\left(\frac{\alpha}{2}\right) & 0 & 0 & 0 \\ 0 & \tan\left(\frac{\alpha}{2}\right) & 0 & 0 \\ 0 & 0 & 0 & -\frac{f-n}{nf} \\ 0 & 0 & 1 & \frac{1}{n} \end{bmatrix}$$

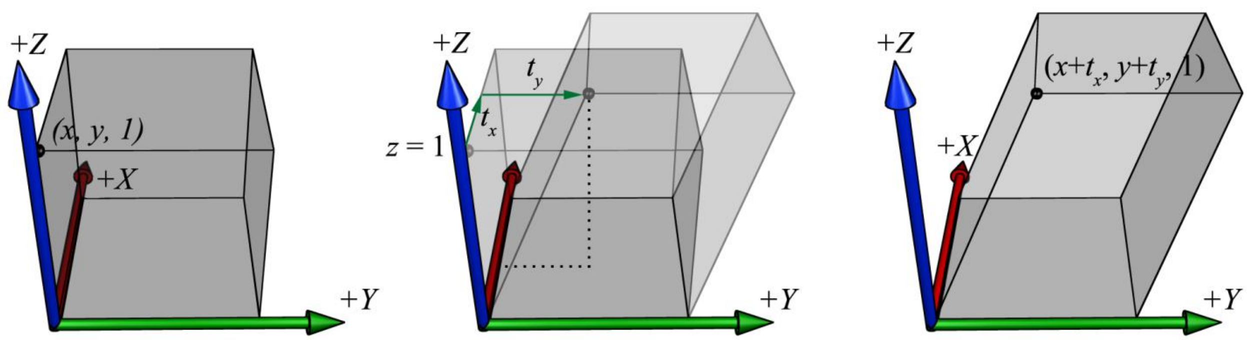
$$= \left[x_{ndc}\, r\tan\left(\frac{\alpha}{2}\right), y_{ndc}\tan\left(\frac{\alpha}{2}\right), z, -z_{ndc}\frac{f-n}{nf}+\frac{1}{n}\right]$$

$$= \left[\frac{zx}{rz\tan(\alpha/2)} r\tan\left(\frac{\alpha}{2}\right), \frac{zy}{z\tan(\alpha/2)}\tan\left(\frac{\alpha}{2}\right), z, -z\left(\frac{f}{f-n}+\frac{-nf}{(f-n)z}\right)\frac{f-n}{nf}+\frac{1}{n}\right]$$

$$= [x, y, z, 1]$$

13. Consider the 3D shear transform given by $S_{xy}(x,y,z) = (x+zt_x, y+zt_y, z)$. This transformation is illustrated in Figure 5.37. Prove that this is a linear transformation and has the following matrix representation:

$$\mathbf{S}_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$



[FigD.11: The *x*- and *y*-coordinates sheared by the *z*-coordinate. The top face of the box lies in the $z = 1$ plane. Observe that the shear transform translates points in this plane.]

Solution:

$$\begin{aligned} S_{xy}(\mathbf{u}+\mathbf{v}) &= \left((u_x+v_x)+(u_z+v_z)t_x, (u_y+v_y)+(u_z+v_z)t_y, u_z+v_z\right) \\ &= \left((u_x+u_zt_x)+(v_x+v_zt_x), (u_y+u_zt_y)+(v_y+v_zt_y), u_z+v_z\right) \\ &= (u_x+u_zt_x, u_y+u_zt_y, u_z) + (v_x+v_zt_x, v_y+v_zt_y, v_z) \\ &= S_{xy}(\mathbf{u}) + S_{xy}(\mathbf{v}) \end{aligned}$$

$$\begin{aligned} S_{xy}(k\mathbf{u}) &= (ku_x+ku_zt_x, ku_y+ku_zt_y, ku_z) \\ &= k(u_x+u_zt_x, u_y+u_zt_y, u_z) \\ &= kS_{xy}(\mathbf{u}) \end{aligned}$$

So S_{xy} is a linear transformation. To find the matrix representation, we apply it to the standard basis vectors:

$$S_{xy}(\mathbf{i}) = S_{xy}(1,0,0) = (1,0,0)$$
$$S_{xy}(\mathbf{j}) = S_{xy}(0,1,0) = (0,1,0)$$
$$S_{xy}(\mathbf{k}) = S_{xy}(0,0,1) = (t_x, t_y, 1)$$

The matrix representation is found by inserting the above three vectors into the rows of a matrix:

$$\mathbf{S}_{xy} = \begin{bmatrix} \leftarrow S_{xy}(\mathbf{i}) \rightarrow \\ \leftarrow S_{xy}(\mathbf{j}) \rightarrow \\ \leftarrow S_{xy}(\mathbf{k}) \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

Chapter 9

5. Let $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ be the vertices of a 3D triangle with respective texture coordinates $\mathbf{q}_0$, $\mathbf{q}_1$, and $\mathbf{q}_2$. Recall from §8.2 that for an arbitrary point on a 3D triangle $\mathbf{p}(s,t) = \mathbf{p}_0 + s(\mathbf{p}_1 - \mathbf{p}_0) + t(\mathbf{p}_2 - \mathbf{p}_0)$ where $s \geq 0, t \geq 0, s + t \leq 1$, its texture coordinates (u, v) are found by linearly interpolating the vertex texture coordinates across the 3D triangle by the same s, t parameters:

$$(u,v) = \mathbf{q}_0 + s(\mathbf{q}_1 - \mathbf{q}_0) + t(\mathbf{q}_2 - \mathbf{q}_0)$$

a) Given (u, v) and $\mathbf{q}_0$, $\mathbf{q}_1$, and $\mathbf{q}_2$, solve for (s,t) in terms of u and v (Hint: Consider the vector equation $(u,v) - \mathbf{q}_0 = s(\mathbf{q}_1 - \mathbf{q}_0) + t(\mathbf{q}_2 - \mathbf{q}_0)$.
b) Express $\mathbf{p}$ as a function of u and v; that is, find a formula $\mathbf{p} = \mathbf{p}(u,v)$.
c) Compute $\partial\mathbf{p}/\partial u$ and $\partial\mathbf{p}/\partial v$ and give a geometric interpretation of what these vectors mean.

Solution:

a) Let $\mathbf{q}_0 = (u_0, v_0)$, $\mathbf{q}_1 = (u_1, v_1)$, and $\mathbf{q}_2 = (u_2, v_2)$:

$$(u,v) - (u_0,v_0) = s(u_1 - u_0, v_1 - v_0) + t(u_2 - u_0, v_2 - v_0)$$

$$\begin{bmatrix} u_1 - u_0 & u_2 - u_0 \\ v_1 - v_0 & v_2 - v_0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

From Example 2.10, the inverse of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is given by:

$$\mathbf{A}^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

We can solve for $\begin{bmatrix} s \\ t \end{bmatrix}$ by multiplying by the inverse:

$$\therefore \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} u_1 - u_0 & u_2 - u_0 \\ v_1 - v_0 & v_2 - v_0 \end{bmatrix}^{-1} \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} v_2 - v_0 & u_0 - u_2 \\ v_0 - v_1 & u_1 - u_0 \end{bmatrix} \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)}$$

In other words, given the texture coordinates (u, v) of a point on a triangle, we can solve for the parametric coordinates (s, t) that yield the 3D point corresponding to the texture point (u, v).

b) Part (a) showed that $s = s(u, v)$ and $t = t(u, v)$. Therefore: $\mathbf{p}(s, t) = \mathbf{p}_0 + s(\mathbf{p}_1 - \mathbf{p}_0) + t(\mathbf{p}_2 - \mathbf{p}_0)$ can be expressed in terms of (u, v):

$$\begin{aligned} \mathbf{p}(u, v) &= \mathbf{p}\big(s(u, v), t(u, v)\big) \\ &= \mathbf{p}_0 + s(u, v)(\mathbf{p}_1 - \mathbf{p}_0) + t(u, v)(\mathbf{p}_2 - \mathbf{p}_0) \end{aligned}$$

c) First, note that:

$$\begin{bmatrix} s(u, v) \\ t(u, v) \end{bmatrix} = \frac{\begin{bmatrix} v_2 - v_0 & u_0 - u_2 \\ v_0 - v_1 & u_1 - u_0 \end{bmatrix} \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix}}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)}$$

$$\begin{bmatrix} \frac{\partial s}{\partial u}(u, v) \\ \frac{\partial t}{\partial u}(u, v) \end{bmatrix} = \frac{\begin{bmatrix} v_2 - v_0 \\ v_0 - v_1 \end{bmatrix}}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)}$$

$$\begin{bmatrix} \frac{\partial s}{\partial v}(u, v) \\ \frac{\partial t}{\partial v}(u, v) \end{bmatrix} = \frac{\begin{bmatrix} u_0 - u_2 \\ u_1 - u_0 \end{bmatrix}}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)}$$

Now the partial derivatives are given by:

$$\begin{aligned} \frac{\partial \mathbf{p}}{\partial u} &= \frac{\partial \mathbf{p}}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial \mathbf{p}}{\partial t}\frac{\partial t}{\partial u} \\ &= (\mathbf{p}_1 - \mathbf{p}_0)\frac{\partial s}{\partial u} + (\mathbf{p}_2 - \mathbf{p}_0)\frac{\partial t}{\partial u} \\ &= \frac{(\mathbf{p}_1 - \mathbf{p}_0)(v_2 - v_0) + (\mathbf{p}_2 - \mathbf{p}_0)(v_0 - v_1)}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)} \\ &= \frac{(\mathbf{p}_1 - \mathbf{p}_0)(v_2 - v_0) - (\mathbf{p}_2 - \mathbf{p}_0)(v_1 - v_0)}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)} \end{aligned}$$

$$= \frac{\Delta v_1 \mathbf{e}_0 - \Delta v_0 \mathbf{e}_1}{\Delta u_0 \Delta v_1 - \Delta u_1 \Delta v_0}$$

where we use the notation:

$$\begin{aligned}
\mathbf{e}_0 &= \mathbf{p}_1 - \mathbf{p}_0 \\
\mathbf{e}_1 &= \mathbf{p}_2 - \mathbf{p}_0 \\
\Delta u_0 &= u_1 - u_0 \\
\Delta u_1 &= u_2 - u_0 \\
\Delta v_0 &= v_1 - v_0 \\
\Delta v_1 &= v_2 - v_0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial \mathbf{p}}{\partial v} &= \frac{\partial \mathbf{p}}{\partial s}\frac{\partial s}{\partial v} + \frac{\partial \mathbf{p}}{\partial t}\frac{\partial t}{\partial v} \\
&= (\mathbf{p}_1 - \mathbf{p}_0)\frac{\partial s}{\partial v} + (\mathbf{p}_2 - \mathbf{p}_0)\frac{\partial t}{\partial v} \\
&= \frac{(\mathbf{p}_1 - \mathbf{p}_0)(u_0 - u_2) + (\mathbf{p}_2 - \mathbf{p}_0)(u_1 - u_0)}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)} \\
&= \frac{-(\mathbf{p}_1 - \mathbf{p}_0)(u_2 - u_0) + (\mathbf{p}_2 - \mathbf{p}_0)(u_1 - u_0)}{(u_1 - u_0)(v_2 - v_0) - (u_2 - u_0)(v_1 - v_0)} \\
&= \frac{-\Delta u_1 \mathbf{e}_0 + \Delta u_0 \mathbf{e}_1}{\Delta u_0 \Delta v_1 - \Delta u_1 \Delta v_0}
\end{aligned}$$

In summary,

$$\frac{\partial \mathbf{p}}{\partial u} = \frac{\Delta v_1 \mathbf{e}_0 - \Delta v_0 \mathbf{e}_1}{\Delta u_0 \Delta v_1 - \Delta u_1 \Delta v_0}$$

$$\frac{\partial \mathbf{p}}{\partial v} = \frac{-\Delta u_1 \mathbf{e}_0 + \Delta u_0 \mathbf{e}_1}{\Delta u_0 \Delta v_1 - \Delta u_1 \Delta v_0}$$

$$\begin{bmatrix} \leftarrow \frac{\partial \mathbf{p}}{\partial u} \rightarrow \\ \leftarrow \frac{\partial \mathbf{p}}{\partial v} \rightarrow \end{bmatrix} = \frac{1}{\Delta u_0 \Delta v_1 - \Delta v_0 \Delta u_1} \begin{bmatrix} \Delta v_1 & -\Delta v_0 \\ -\Delta u_1 & \Delta u_0 \end{bmatrix} \begin{bmatrix} \leftarrow \mathbf{e}_0 \rightarrow \\ \leftarrow \mathbf{e}_1 \rightarrow \end{bmatrix}$$

Compare this result to the one derived in §19.3; we have simple derived the forumla in §19.3 in a different way.

The 3D vector $\frac{\partial \mathbf{p}}{\partial u}$ gives us the "velocity" we move in 3D space when we move in the *u*-direction in texture space. Put another way, it tells us the direction of the texture space *u*-axis in 3D space. Likewise, the 3D vector $\frac{\partial \mathbf{p}}{\partial v}$ gives us the "velocity" we move in 3D space when we move in the *v*-direction in texture space (i.e., the direction of the texture space *v*-axis in 3D space).

6. See "TexColumns" on the DVD.

Chapter 11

2. Prove that $\mathbf{s} = \mathbf{p} - \frac{\mathbf{n}\cdot\mathbf{p}+d}{\mathbf{n}\cdot(\mathbf{p}-\mathbf{L})}(\mathbf{p}-\mathbf{L}) = \mathbf{pS}_{point}$ by doing the matrix multiplication for each component, as was done in §10.5.1 for directional lights.

Solution:

Let $\mathbf{p} = (p_1, p_2, p_3, 1)$. For $i \in \{1,2,3\}$, the *i*th coordinate of $\mathbf{s} = \mathbf{pS}_{point}$ is given by:

$$
\begin{aligned}
s_i' &= (\mathbf{n}\cdot\mathbf{L} + d)p_i - L_i(\mathbf{n}\cdot\mathbf{p} + d) \\
&= p_i\mathbf{n}\cdot\mathbf{L} + p_i d - L_i\mathbf{n}\cdot\mathbf{p} - L_i d \\
&= p_i\mathbf{n}\cdot\mathbf{L} + p_i d + (p_i\mathbf{n}\cdot\mathbf{p} - p_i\mathbf{n}\cdot\mathbf{p}) - L_i\mathbf{n}\cdot\mathbf{p} - L_i d \\
&= p_i\mathbf{n}\cdot\mathbf{L} - p_i\mathbf{n}\cdot\mathbf{p} + p_i(\mathbf{n}\cdot\mathbf{p} + d) - L_i(\mathbf{n}\cdot\mathbf{p} + d) \\
&= p_i\mathbf{n}\cdot(\mathbf{L} - \mathbf{p}) + p_i(\mathbf{n}\cdot\mathbf{p} + d) - L_i(\mathbf{n}\cdot\mathbf{p} + d) \\
&= -p_i\mathbf{n}\cdot(\mathbf{p} - \mathbf{L}) + (p_i - L_i)(\mathbf{n}\cdot\mathbf{p} + d)
\end{aligned}
$$

and the fourth coordinate is given by:

$$
\begin{aligned}
s_4' &= -\mathbf{n}\cdot\mathbf{p} + \mathbf{n}\cdot\mathbf{L} \\
&= -\mathbf{n}\cdot(\mathbf{p} - \mathbf{L})
\end{aligned}
$$

Doing the homogeneous divide we obtain:

$$
s_i'' = \frac{-p_i\mathbf{n}\cdot(\mathbf{p} - \mathbf{L}) + (p_i - L_i)(\mathbf{n}\cdot\mathbf{p} + d)}{-\mathbf{n}\cdot(\mathbf{p} - \mathbf{L})}
$$

$$
= p_i - \frac{\mathbf{n}\cdot\mathbf{p} + d}{\mathbf{n}\cdot(\mathbf{p} - \mathbf{L})}(p_i - L_i)
$$

But this is exactly the *i*th coordinate of $\mathbf{s} = \mathbf{p} - \frac{\mathbf{n}\cdot\mathbf{p}+d}{\mathbf{n}\cdot(\mathbf{p}-\mathbf{L})}(\mathbf{p}-\mathbf{L})$, so

$$
\mathbf{s} = \mathbf{p} - \frac{\mathbf{n}\cdot\mathbf{p} + d}{\mathbf{n}\cdot(\mathbf{p} - \mathbf{L})}(\mathbf{p} - \mathbf{L}) = \mathbf{pS}_{point}
$$

Chapter 13

5. See "WavesCS" on the DVD.

6. See “SobelFilter” on the DVD.

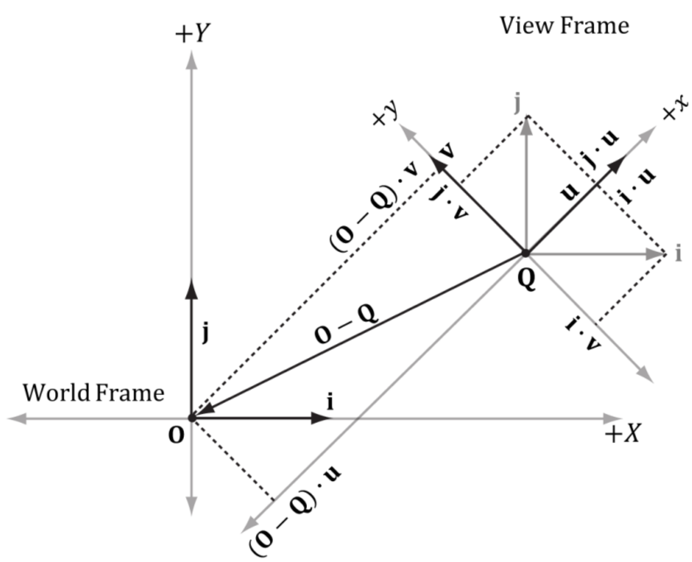
Chapter 15

1. Given the world space axes and origin in world coordinates: $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ and $\mathbf{O} = (0,0,0)$, and the view space axes and origin in world coordinates: $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, $\mathbf{w} = (w_x, w_y, w_z)$ and $\mathbf{Q} = (Q_x, Q_y, Q_z)$, derive the view matrix form

$$\mathbf{V} = \begin{bmatrix} u_x & v_x & w_x & 0 \\ u_y & v_y & w_y & 0 \\ u_z & v_z & w_z & 0 \\ -\mathbf{Q}\cdot\mathbf{u} & -\mathbf{Q}\cdot\mathbf{v} & -\mathbf{Q}\cdot\mathbf{w} & 1 \end{bmatrix}$$

using the dot product. (Remember, to find the change of coordinate matrix from world space to view space, you just need to describe the world space axes and origin with coordinates relative to view space. Then these coordinates become the rows of the view matrix.)

Solution:



[Figure D.12: Finding the world space coordinates in view space.]

All the given vectors have coordinates in world space, and all the axis vectors are unit vectors. However, using the dot product, we can obtain view space coordinates of the world space. From the figure, we see:

$$[\mathbf{i}]_V = (\mathbf{i}\cdot\mathbf{u}, \mathbf{i}\cdot\mathbf{v}, \mathbf{i}\cdot\mathbf{w}) = (u_x, v_x, w_x)$$
$$[\mathbf{j}]_V = (\mathbf{j}\cdot\mathbf{u}, \mathbf{j}\cdot\mathbf{v}, \mathbf{j}\cdot\mathbf{w}) = (u_y, v_y, w_y)$$
$$[\mathbf{k}]_V = (\mathbf{k}\cdot\mathbf{u}, \mathbf{k}\cdot\mathbf{v}, \mathbf{k}\cdot\mathbf{w}) = (u_z, v_z, w_z)$$

$$[\mathbf{O}]_V = \big((\mathbf{O}-\mathbf{Q})\cdot\mathbf{u}, (\mathbf{O}-\mathbf{Q})\cdot\mathbf{v}, (\mathbf{O}-\mathbf{Q})\cdot\mathbf{w}\big) = (-\mathbf{Q}\cdot\mathbf{u}, -\mathbf{Q}\cdot\mathbf{v}, -\mathbf{Q}\cdot\mathbf{w})$$

Augmenting to homogeneous coordinates and putting these vectors into the rows of a matrix yields the view matrix.

Chapter 16

2. The plane equations in NDC space take on a very simple form. All points inside the view frustum are bounded as follows:

$$\begin{gathered} -1 \le x_{ndc} \le 1 \\ -1 \le y_{ndc} \le 1 \\ 0 \le z_{ndc} \le 1 \end{gathered}$$

In particular, the left plane equation is given by $x = -1$ and the right plane equation is given by $x = 1$ in NDC space. In homogeneous clip space before the perspective divide, all points inside the view frustum are bounded as follows:

$$\begin{gathered} -w \le x_h \le w \\ -w \le y_h \le w \\ 0 \le z_h \le w \end{gathered}$$

Here, the left plane is defined by $w = -x_h$ and the right plane is defined by $w = x_h$. Let $\mathbf{M} = \mathbf{VP}$ be the view-projection matrix product, and let $\mathbf{v} = (x, y, z, 1)$ be a point in world space inside the frustum. Consider $(x_h, y_h, z_h, w) = \mathbf{vM} = \big(\mathbf{v}\cdot\mathbf{M}_{*,1}, \mathbf{v}\cdot\mathbf{M}_{*,2}, \mathbf{v}\cdot\mathbf{M}_{*,3}, \mathbf{v}\cdot\mathbf{M}_{*,4}\big)$ to show that the inward facing frustum planes in world space are given by:

Left	$0 = \mathbf{p}\cdot\big(\mathbf{M}_{*,1} + \mathbf{M}_{*,4}\big)$
Right	$0 = \mathbf{p}\cdot\big(\mathbf{M}_{*,4} - \mathbf{M}_{*,1}\big)$
Bottom	$0 = \mathbf{p}\cdot\big(\mathbf{M}_{*,2} + \mathbf{M}_{*,4}\big)$
Top	$0 = \mathbf{p}\cdot\big(\mathbf{M}_{*,4} - \mathbf{M}_{*,2}\big)$
Near	$0 = \mathbf{p}\cdot\mathbf{M}_{*,3}$
Far	$0 = \mathbf{p}\cdot\big(\mathbf{M}_{*,4} - \mathbf{M}_{*,3}\big)$

[Notes]

a) We ask for inward facing normals. That means a point inside the frustum has a positive distance from the plane; in other words, $\mathbf{n}\cdot\mathbf{p} + d \ge 0$ for a point $\mathbf{p}$ inside the frustum.

b) Note that $p_w = 1$, so the above dot product formulas do yield plane equations of the form $Ax + By + Cz + D = 0$.

c) The calculated plane normal vectors are not unit length; see Appendix C for how to normalize a plane.

[/Notes]

Solution:

We assume $\mathbf{v} = (x, y, z, 1)$ is a point inside the frustum. Therefore, its coordinates $(x_h, y_h, z_h, w) = \mathbf{vM} = \left(\mathbf{v} \cdot \mathbf{M}_{*,1}, \mathbf{v} \cdot \mathbf{M}_{*,2}, \mathbf{v} \cdot \mathbf{M}_{*,3}, \mathbf{v} \cdot \mathbf{M}_{*,4}\right)$ in homogeneous space are bounded as follows:

$$\begin{gathered} -w \le x_h \le w \\ -w \le y_h \le w \\ 0 \le z_h \le w \end{gathered}$$

We can make substitutions to find the plane bounds in world space.

Left Plane:

$$\begin{aligned} -w &\le x_h \\ -\mathbf{v} \cdot \mathbf{M}_{*,4} &\le \mathbf{v} \cdot \mathbf{M}_{*,1} \\ 0 &\le \mathbf{v} \cdot \left(\mathbf{M}_{*,1} + \mathbf{M}_{*,4}\right) \end{aligned}$$

That is, $\mathbf{v}$ is in the positive half-space of the world space plane $\mathbf{p} \cdot \left(\mathbf{M}_{*,1} + \mathbf{M}_{*,4}\right) = 0$.

Right Plane:

$$\begin{aligned} x_h &\le w \\ \mathbf{v} \cdot \mathbf{M}_{*,1} &\le \mathbf{v} \cdot \mathbf{M}_{*,4} \\ 0 &\le \mathbf{v} \cdot \left(\mathbf{M}_{*,4} - \mathbf{M}_{*,1}\right) \end{aligned}$$

That is, $\mathbf{v}$ is in the positive half-space of the world space plane $\mathbf{p} \cdot \left(\mathbf{M}_{*,4} - \mathbf{M}_{*,1}\right) = 0$.

Bottom Plane:

$$\begin{aligned} -w &\le y_h \\ -\mathbf{v} \cdot \mathbf{M}_{*,4} &\le \mathbf{v} \cdot \mathbf{M}_{*,2} \\ 0 &\le \mathbf{v} \cdot \left(\mathbf{M}_{*,2} + \mathbf{M}_{*,4}\right) \end{aligned}$$

That is, $\mathbf{v}$ is in the positive half-space of the world space plane $\mathbf{p} \cdot \left(\mathbf{M}_{*,2} + \mathbf{M}_{*,4}\right) = 0$.

Top Plane:

$$\begin{aligned} y_h &\le w \\ \mathbf{v} \cdot \mathbf{M}_{*,2} &\le \mathbf{v} \cdot \mathbf{M}_{*,4} \\ 0 &\le \mathbf{v} \cdot \left(\mathbf{M}_{*,4} - \mathbf{M}_{*,2}\right) \end{aligned}$$

That is, $\mathbf{v}$ is in the positive half-space of the world space plane $\mathbf{p} \cdot \left(\mathbf{M}_{*,4} - \mathbf{M}_{*,2}\right) = 0$.

Near Plane:

$$0 \le z_h$$
$$0 \le \mathbf{v} \cdot \mathbf{M}_{*,3}$$

That is, $\mathbf{v}$ is in the positive half-space of the world space plane $\mathbf{p} \cdot \mathbf{M}_{*,3} = 0$.

Far Plane:

$$\begin{aligned} z_h &\le w \\ \mathbf{v} \cdot \mathbf{M}_{*,3} &\le \mathbf{v} \cdot \mathbf{M}_{*,4} \\ 0 &\le \mathbf{v} \cdot \left(\mathbf{M}_{*,4} - \mathbf{M}_{*,3}\right) \end{aligned}$$

That is, $\mathbf{v}$ is in the positive half-space of the world space plane $\mathbf{p} \cdot \left(\mathbf{M}_{*,4} - \mathbf{M}_{*,3}\right) = 0$.

4. An OBB can be defined by a center point $\mathbf{C}$, three orthonormal axis vectors $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$ defining the box orientation, and three extent lengths a_0, a_1, and a_2 along the box axes $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$, respectivey, that give the distance from the box center to the box sides.

a) Consider Figure 15.13 (which shows the situation in 2D) and conclude the projected "shadow" of the OBB onto the axis defined by the normal vector is $2r$, where

$$r = |a_0\mathbf{r}_0 \cdot \mathbf{n}| + |a_1\mathbf{r}_1 \cdot \mathbf{n}| + |a_2\mathbf{r}_2 \cdot \mathbf{n}|$$

b) In the previous formula for r, explain why we must take the absolute values instead of just computing $r = (a_0\mathbf{r}_0 + a_1\mathbf{r}_1 + a_2\mathbf{r}_2) \cdot \mathbf{n}$?

c) Derive a plane/OBB intersection test that determines if the OBB is in front of the plane, behind the plane, or intersecting the plane.

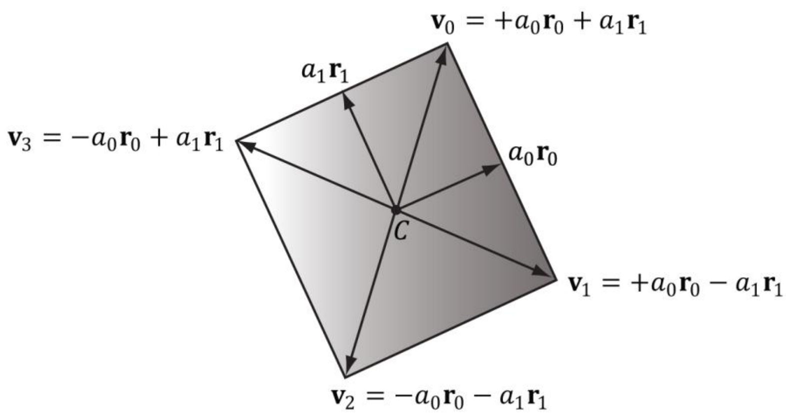
d) An AABB is a special case of an OBB, so this test also works for an AABB. However, the formula for r simplifies in the case of an AABB. Find the simplified formula for r for the AABB case.

Solution:

b) If one of the $\mathbf{r}_i \cdot \mathbf{n}$ terms is negative, the sum $(a_0\mathbf{r}_0 + a_1\mathbf{r}_1 + a_2\mathbf{r}_2) \cdot \mathbf{n}$ will not give the "radius" of the OBB. To explain it another way (in 2D), let the vectors from the box center to the corners be given by:

$$\begin{aligned} \mathbf{v}_0 &= +a_0\mathbf{r}_0 + a_1\mathbf{r}_1 \\ \mathbf{v}_1 &= +a_0\mathbf{r}_0 - a_1\mathbf{r}_1 \\ \mathbf{v}_2 &= -a_0\mathbf{r}_0 - a_1\mathbf{r}_1 \end{aligned}$$

$$\mathbf{v}_3 = -a_0\mathbf{r}_0 + a_1\mathbf{r}_1$$



[Figure D.13: The corner vectors. Note that these are not the corner points, but the vectors from the center point to the corner points.]

The OBB "radius" is the corner vector that gives the largest projection onto **n**:

$$r = \max(\mathbf{n}\cdot\mathbf{v}_0, \mathbf{n}\cdot\mathbf{v}_1, \mathbf{n}\cdot\mathbf{v}_2, \mathbf{n}\cdot\mathbf{v}_3)$$

$$\begin{aligned}\mathbf{n}\cdot\mathbf{v}_0 &= \mathbf{n}\cdot a_0\mathbf{r}_0 + \mathbf{n}\cdot a_1\mathbf{r}_1\\ \mathbf{n}\cdot\mathbf{v}_1 &= \mathbf{n}\cdot a_0\mathbf{r}_0 - \mathbf{n}\cdot a_1\mathbf{r}_1\\ \mathbf{n}\cdot\mathbf{v}_2 &= -\mathbf{n}\cdot a_0\mathbf{r}_0 - \mathbf{n}\cdot a_1\mathbf{r}_1\\ \mathbf{n}\cdot\mathbf{v}_3 &= -\mathbf{n}\cdot a_0\mathbf{r}_0 + \mathbf{n}\cdot a_1\mathbf{r}_1\end{aligned}$$

The maximum will be the one where all terms are positive (or all terms are negative) so that there is no cancellation between the terms. In other words, the maximum will be equal to:

$$r = \max(\mathbf{n}\cdot\mathbf{v}_0, \mathbf{n}\cdot\mathbf{v}_1, \mathbf{n}\cdot\mathbf{v}_2, \mathbf{n}\cdot\mathbf{v}_3) = |a_0\mathbf{r}_0\cdot\mathbf{n}| + |a_1\mathbf{r}_1\cdot\mathbf{n}|$$

The same argument generalizes to 3D where there are 8 corner vectors.

c) The signed distance from the center of the OBB to the plane is $k = \mathbf{n}\cdot\mathbf{c} + d$. If $|k| \le r$ then the sphere intersects the plane. If $k < -r$ then the OBB is behind the plane. If $k > r$ then the OBB is in front of the plane and the sphere intersects the positive half-space of the plane.

d) In the case of an AABB, $\mathbf{r}_0 = (1,0,0)$, $\mathbf{r}_1 = (0,1,0)$, and $\mathbf{r}_2 = (0,0,1)$; therefore:

$$\begin{aligned}r &= |a_0\mathbf{r}_0\cdot\mathbf{n}| + |a_1\mathbf{r}_1\cdot\mathbf{n}| + |a_2\mathbf{r}_2\cdot\mathbf{n}|\\ &= |a_0 n_x| + |a_1 n_y| + |a_2 n_z|\end{aligned}$$

Chapter 20

6. Derive the matrix that maps the box $[l,r]\times[b,t]\times[n,f]\to[-1,1]\times[-1,1]\times[0,1]$. This is an "off center" orthographic view volume (i.e., the box is not centered about the view space

origin). In contrast, the orthographic projection matrix derived in §21.2 is an "on center" orthographic view volume.

Solution:

For all three coordinates, we need to remap an interval onto another interval. We can solve this problem generally once, and then apply it to each coordinate. We want to map $[s,t] \rightarrow [u,v]$. We assume the mapping takes the form $g(x) = ax + b$ (i.e., a scaling and translation). We have the conditions $g(s) = u$ and $g(t) = v$, which allow us to solve for a and b:

$$\begin{aligned} as + b &= u \\ at + b &= v \end{aligned}$$

The first equation implies $b = u - as$. Plugging this into the second equation we get:

$$\begin{aligned} at + u - as &= v \\ a(t-s) &= v - u \\ a &= \frac{v-u}{t-s} \end{aligned}$$

And so:

$$\begin{aligned} b &= u - as \\ &= u - \frac{v-u}{t-s}s \\ &= \frac{u(t-s) - vs + us}{t-s} \\ &= \frac{ut - us - vs + us}{t-s} \\ &= \frac{ut - vs}{t-s} \end{aligned}$$

Therefore,

$$g(x) = \frac{v-u}{t-s}x + \frac{ut-vs}{t-s}$$

Applying this formula to our specific intervals, we obtain the transformations:

$$[l,r] \rightarrow [-1,1]$$

$$x' = \frac{2}{r-l}x + \frac{l-r}{r-l}$$

$$[t,b] \rightarrow [-1,1]$$

$$y' = \frac{2}{t-b}y + \frac{b-t}{t-b}$$

$$[n, f] \rightarrow [0,1]$$

$$z' = \frac{1}{f-n}z + \frac{-n}{f-n}$$

Or in terms of matrices:

$$[x', y', z', 1] = [x, y, z, 1]\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{1}{f-n} & 0 \\ \frac{l-r}{r-l} & \frac{b-t}{t-b} & \frac{-n}{f-n} & 1 \end{bmatrix}$$

7. In Chapter 17 we learned about picking with a perspective projection matrix. Derive picking formulas for an off-centered orthographic projection.

Solution:

Let (s_x, s_y) be the picked point in screen space. Inverting the viewport transformation, we get the corresponding point in NDC space:

$$x_{ndc} = \frac{2s_x}{w} - 1$$
$$y_{ndc} = -\frac{2s_y}{h} + 1$$

The orthographic projection matrix transforms the view volume from view space to NDC space. The off-centered orthographic projection transformation is:

$$[x_v, y_v, z_v, 1]\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{1}{f-n} & 0 \\ \frac{l-r}{r-l} & \frac{b-t}{t-b} & \frac{-n}{f-n} & 1 \end{bmatrix} = [x_{ndc}, y_{ndc}, z_{ndc}, 1]$$

In particular, this gives the two equations:

$$\frac{2x_v}{r-l}+\frac{l-r}{r-l}=x_{ndc}$$
$$\frac{2y_v}{t-b}+\frac{b-t}{t-b}=y_{ndc}$$

We can then solve for the view space coordinates (x_v, y_v) in terms of (s_x, s_y):

$$\begin{aligned}
\frac{2x_v}{r-l}+\frac{l-r}{r-l}&=x_{ndc}\\
\frac{2x_v}{r-l}&=x_{ndc}-\frac{l-r}{r-l}\\
x_v&=\frac{r-l}{2}x_{ndc}-\frac{r-l}{2}\cdot\frac{l-r}{r-l}\\
x_v&=\frac{x_{ndc}-\mathbf{P}_{30}}{\mathbf{P}_{00}}\\
x_v&=\frac{\frac{2s_x}{w}-1-\mathbf{P}_{30}}{\mathbf{P}_{00}}
\end{aligned}$$

$$\begin{aligned}
\frac{2y_v}{t-b}+\frac{b-t}{t-b}&=y_{ndc}\\
\frac{2y_v}{t-b}&=y_{ndc}-\frac{b-t}{t-b}\\
y_v&=\frac{t-b}{2}y_{ndc}-\frac{t-b}{2}\cdot\frac{b-t}{t-b}\\
y_v&=\frac{y_{ndc}-\mathbf{P}_{31}}{\mathbf{P}_{11}}\\
y_v&=\frac{-\frac{2s_y}{h}+1-\mathbf{P}_{31}}{\mathbf{P}_{11}}
\end{aligned}$$

Then our picking ray in view space is given by $\mathbf{r}(t)=(x_v, y_v, 0)+t(0,0,1)$. Note that in an orthographic projection, all the rays are parallel to the *z*-axis direction $(0,0,1)$.

Sample code:

```
Ray3 CalcWorldPickRay(XMMATRIX P, Point s, Size viewport)
{
	float sx = (float)s.X;
	float sy = (float)s.Y;

	float w = (float)viewport.Width;
	float h = (float)viewport.Height;

	float x = (2.0f*sx/w - 1.0f)/P(0,0)  - P(3,0)/P(0,0);
	float y = (-2.0f*sy/h + 1.0f)/P(1,1) - P(3,1)/P(1,1);

	Ray3 pickRay;
	pickRay.Origin    = XMFLOAT3(x, y, 0.0f);
	pickRay.Direction = XMFLOAT3(0.0f, 0.0f, 1.0f);

	return pickRay;
}
```

Note: This works for an off-centered orthographic projection matrix and a centered one since a centered one is just a special case of the off-centered one.

Chapter 22

3. Rotate the vector $(2, 1)$ 30° using complex number multiplication.

Solution:

$$\mathbf{z} = 2 + i$$
$$\mathbf{z}_2 = (\cos 30° + i \sin 30°) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\begin{aligned}
\mathbf{z}' = \mathbf{z}\mathbf{z}_2 &= (2+i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\
&= \sqrt{3} + i + \frac{\sqrt{3}}{2}i - \frac{1}{2} \\
&= \frac{2\sqrt{3}-1}{2} + \frac{\sqrt{3}+2}{2}i
\end{aligned}$$

We can verify this by seeing if we get the same answer with a rotation matrix:

$$[2,1]\begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix} = [2,1]\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \left[\frac{2\sqrt{3}-1}{2}, \frac{\sqrt{3}+2}{2}\right]$$

5. Let $\mathbf{z} = a + ib$. Show $|\mathbf{z}|^2 = \mathbf{z}\bar{\mathbf{z}}$.

Solution:

$$\begin{aligned}
\mathbf{z}\bar{\mathbf{z}} &= (a+ib)(a-ib) \\
&= a^2 - abi + abi - b^2i^2 \\
&= a^2 + b^2 \\
&= \left(\sqrt{a^2+b^2}\right)^2 \\
&= |\mathbf{z}|^2
\end{aligned}$$

6. Let $\mathbf{M}$ be a 2×2 matrix. Prove that $\det \mathbf{M} = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^T$ if and only if $\mathbf{M} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. That is, if and only if $\mathbf{M}$ is a rotation matrix. This gives us a way of testing if a matrix is a rotation matrix.

Solution:

Let $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and suppose $\det \mathbf{M} = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^T$.

$\det \mathbf{M} = 1$ along with

$$\mathbf{M}\mathbf{M}^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

imply the following equations:

$$\begin{gathered} M_{11}^2 + M_{12}^2 = 1 \\ M_{21}^2 + M_{22}^2 = 1 \\ M_{11}M_{22} - M_{12}M_{21} = 1 \end{gathered}$$

Now,

$$\begin{gathered} (M_{11}^2 + M_{12}^2) + (M_{21}^2 + M_{22}^2) = 2 \\ 2(M_{11}M_{22} - M_{12}M_{21}) = 2 \end{gathered}$$

Therefore,

$$\begin{aligned} M_{11}^2 + M_{12}^2 + M_{21}^2 + M_{22}^2 - 2M_{11}M_{22} + 2M_{12}M_{21} &= 0 \\ (M_{11}^2 - 2M_{11}M_{22} + M_{22}^2) + (M_{12}^2 + 2M_{12}M_{21} + M_{21}^2) &= 0 \\ (M_{11} - M_{22})^2 + (M_{12} + M_{21})^2 &= 0 \end{aligned}$$

Both terms are positive; hence, we must have:

$$M_{11} = M_{22}$$

and

$$M_{21} = -M_{12}$$

So our matrix has the form:

$$\mathbf{M} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

Finally, because $\det \mathbf{M} = A^2 + B^2 = 1$ there exists a θ such that $A = \cos\theta$ and $B = \sin\theta$, and we get our result:

$$\mathbf{M} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now suppose $\mathbf{M} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Then

$$\det \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\mathbf{M}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{M}\mathbf{M}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{M}^T\,\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, $\det \mathbf{M} = 1$ and $\mathbf{M}^{-1} = \mathbf{M}^T$. In other words, all rotation matrices have determinant 1 and are orthogonal, and every orthogonal matrix with determinant 1 must be a rotation matrix.

7. Let $\mathbf{p} = (1, 2, 3, 4)$ and $\mathbf{q} = (2, -1, 1, -2)$ be quaternions. Perform the indicated quaternion operations.

a. $\mathbf{p} + \mathbf{q}$
b. $\mathbf{p} - \mathbf{q}$
c. $\mathbf{pq}$
d. $\mathbf{p}^*$
e. $\mathbf{q}^*$
f. $\mathbf{p}^*\mathbf{p}$
g. $\|\mathbf{p}\|$
h. $\|\mathbf{q}\|$
i. $\mathbf{p}^{-1}$
j. $\mathbf{q}^{-1}$

Solution:

a. $\mathbf{p} + \mathbf{q} = (3, 1, 4, 2)$
b. $\mathbf{p} - \mathbf{q} = (-1, 3, 2, 6)$
c. $\mathbf{pq} = \begin{bmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 2 & 1 \\ 3 & 4 & -1 & 2 \\ -2 & 1 & 4 & 3 \\ -1 & -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \\ -7 \\ -11 \end{bmatrix}$
d. $\mathbf{p}^* = (-1, -2, -3, 4)$
e. $\mathbf{q}^* = (-2, 1, -1, -2)$
f. $\mathbf{p}^*\mathbf{p} = \|\mathbf{p}\|^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$
g. $\|\mathbf{p}\| = \sqrt{30}$
h. $\|\mathbf{q}\| = \sqrt{2^2 + 1^2 + 1^2 + 2^2} = \sqrt{10}$
i. $\mathbf{p}^{-1} = \frac{\mathbf{p}^*}{\|\mathbf{p}\|^2} = \frac{(-1,-2,-3,4)}{30}$
j. $\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2} = \frac{(-2,1,-1,-2)}{10}$

```
XMVECTOR p = XMVectorSet(1.0f, 2.0f, 3.0f, 4.0f);
XMVECTOR q = XMVectorSet(2.0f, -1.0f, 1.0f, -2.0f);

XMVECTOR a = p + q;
XMVECTOR b = p - q;
XMVECTOR c = XMQuaternionMultiply(q,p );
XMVECTOR d = XMQuaternionConjugate(p);
```

```
XMVECTOR e = XMQuaternionConjugate(q);
XMVECTOR f = XMQuaternionMultiply(XMQuaternionConjugate(p),p);
XMVECTOR g = XMQuaternionLength(p);
XMVECTOR h = XMQuaternionLength(q);
XMVECTOR i = XMQuaternionInverse(p);
XMVECTOR j = XMQuaternionInverse(q);

cout << "p + q = " << a << endl;
cout << "p - q = " << b  << endl;
cout << "pq = " << c << endl;
cout << "p* = " << d << endl;
cout << "q* = " << e << endl;
cout << "p*p = " << f << endl;
cout << "||p|| = " << g << endl;
cout << "||q|| = " << h << endl;
cout << "invP = " << i << endl;
cout << "invQ = " << j << endl;
```

9. Write the unit quaternion $\mathbf{q} = \left(\frac{\sqrt{3}}{2}, 0, 0, -\frac{1}{2}\right)$ in polar notation.

$$w = \cos\theta \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°$$

$$\mathbf{n} = \frac{\left(\frac{\sqrt{3}}{2}, 0, 0\right)}{\sin(120°)} = \frac{\left(\frac{\sqrt{3}}{2}, 0, 0\right)}{\frac{\sqrt{3}}{2}} = (1, 0, 0)$$

$$\mathbf{q} = (\sin(120°)\,(1, 0, 0), \cos(120°))$$

10. Find the unit quaternion that rotates $45°$ about the axis $(1, 1, 1)$.

$$\begin{aligned}\mathbf{q} &= \left(\sin\left(\frac{\theta}{2}\right)\mathbf{n}, \cos\left(\frac{\theta}{2}\right)\right) \\ &= \left(\sin(22.5°)\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \cos(22.5°)\right) \\ &= (.22094, .22094, .22094, .92388)\end{aligned}$$

The division by 2 is to compensate for the 2θ in Equation 24.3 because we want to rotate by the angle θ, not 2θ.

```
XMVECTOR p = XMQuaternionRotationAxis(
      XMVectorSet(1.0f, 1.0f, 1.0f, 0.0f),
      XMConvertToRadians(45.0f));
cout << "p = " << p << endl;
```

11. Find the unit quaternion that rotates $60°$ about the axis $(0, 0, -1)$.

$$\begin{aligned}\mathbf{q} &= \left(\sin\left(\frac{\theta}{2}\right)\mathbf{n}, \cos\left(\frac{\theta}{2}\right)\right)\\ &= (\sin(30°)\,(0,0,-1), \cos(30°))\\ &= \left(0,0,-\frac{1}{2},\frac{\sqrt{3}}{2}\right)\end{aligned}$$

The division by 2 is to compensate for the 2θ in Equation 24.3 because we want to rotate by the angle θ, not 2θ.

```
XMVECTOR p = XMQuaternionRotationAxis(
      XMVectorSet(0.0f, 0.0f, -1.0f, 0.0f),
      XMConvertToRadians(60.0f));
cout << "p = " << p << endl;
```

12. Let $\mathbf{p} = \left(\frac{1}{2},0,0,\frac{\sqrt{3}}{2}\right)$ and $\mathbf{q} = \left(\frac{\sqrt{3}}{2},0,0,\frac{1}{2}\right)$. Compute slerp $\left(\mathbf{p},\mathbf{q},\frac{1}{2}\right)$ and verify it is a unit quaternion.

Solution:

It is easy to verify that $\|\mathbf{p}\| = \|\mathbf{q}\| = 1$. The angle between the quaternions is given by:

$$\theta = \cos^{-1}(\mathbf{p}\cdot\mathbf{q}) = \cos^{-1}\left(\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°$$

$$\text{slerp}(\mathbf{a},\mathbf{b},t) = \frac{\sin\big((1-t)\theta\big)\,\mathbf{a} + \sin(t\theta)\,\mathbf{b}}{\sin\theta}$$

$$\begin{aligned}\mathbf{r} = \text{slerp}\left(\mathbf{p},\mathbf{q},\frac{1}{2}\right) &= \frac{\sin(15°)\,\mathbf{p} + \sin(15°)\,\mathbf{q}}{\sin 30°}\\ &= \frac{\frac{\sqrt{6}-\sqrt{2}}{4}\mathbf{p} + \frac{\sqrt{6}-\sqrt{2}}{4}\mathbf{q}}{\frac{1}{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{2}\left(\frac{1}{2},0,0,\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{6}-\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2},0,0,\frac{1}{2}\right)\\ &= \left(\frac{\sqrt{6}-\sqrt{2}}{4},0,0,\frac{3\sqrt{2}-\sqrt{6}}{4}\right) + \left(\frac{3\sqrt{2}-\sqrt{6}}{4},0,0,\frac{\sqrt{6}-\sqrt{2}}{4}\right)\\ &= \left(\frac{2\sqrt{2}}{4},0,0,\frac{2\sqrt{2}}{4}\right)\\ &= \left(\frac{\sqrt{2}}{2},0,0,\frac{\sqrt{2}}{2}\right)\end{aligned}$$

The interpolated quaternion is easily seen to be unit length: $\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = 1$.

Observe that $R_{\mathbf{p}}$ rotates about the axis $(1,0,0)$ by an angle $2\theta = 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 2 \cdot 30° = 60°$, that $R_{\mathbf{q}}$ rotates about the axis $(1,0,0)$ by an angle $2\theta = 2\cos^{-1}\left(\frac{1}{2}\right) = 2 \cdot 60° = 120°$, and that $R_{\mathbf{r}}$ rotates about the axis $(1,0,0)$ by an angle $2\theta = 2\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 2 \cdot 45° = 90°$. With the interpolation parameter being the midpoint $\frac{1}{2}$, this makes sense, as $90°$ is right in the middle between $60°$ and $120°$.

```
XMVECTOR p = XMVectorSet(0.5f, 0.0f, 0.0f, sqrt(3.0)/2.0f);
XMVECTOR q = XMVectorSet(sqrt(3.0)/2.0f, 0.0f, 0.0f, 0.5f);

XMVECTOR r = XMQuaternionSlerp(p, q, 0.5f);
cout << "r = " << r << endl;
```

14. Prove that $\mathbf{q}\mathbf{q}^* = \mathbf{q}^*\mathbf{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2 = \|\mathbf{u}\|^2 + q_4^2$

Solution:

Let $\mathbf{q} = (\mathbf{u}, w) = (q_1, q_2, q_3, q_4)$.

$$\begin{aligned}
\mathbf{q}\mathbf{q}^* &= (\mathbf{u}, w)(-\mathbf{u}, w) \\
&= (-w\mathbf{u} + w\mathbf{u} + \mathbf{u} \times (-\mathbf{u}), w^2 + \mathbf{u} \cdot \mathbf{u}) \\
&= (\mathbf{0}, \|\mathbf{u}\|^2 + w^2) \\
&= \|\mathbf{u}\|^2 + w^2 \\
&= q_1^2 + q_2^2 + q_3^2 + q_4^2
\end{aligned}$$

Because the quaternion has zero vector part, we convert it to a real number as discussed in §24.2.4. Similarly, it can be shown $\mathbf{q}^*\mathbf{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2$.

16. Prove the following properties:

a. $(\mathbf{p}\mathbf{q})^* = \mathbf{q}^*\mathbf{p}^*$
b. $(\mathbf{p} + \mathbf{q})^* = \mathbf{p}^* + \mathbf{q}^*$
c. $(s\mathbf{q})^* = s\mathbf{q}^*$ for $s \in \mathbb{R}$
d. $\|\mathbf{p}\mathbf{q}\| = \|\mathbf{p}\|\|\mathbf{q}\|$

Solution:

Let $\mathbf{p} = (\mathbf{u}, a)$ and $\mathbf{q} = (\mathbf{v}, b)$ so that $\mathbf{p}^* = (-\mathbf{u}, a)$ and $\mathbf{q}^* = (-\mathbf{v}, b)$.

a)

$$\begin{aligned}
\mathbf{q}^*\mathbf{p}^* &= (-\mathbf{v}, b)(-\mathbf{u}, a) = (b(-\mathbf{u}) + a(-\mathbf{v}) + (-\mathbf{v}) \times (-\mathbf{u}), ba - (-\mathbf{v}) \cdot (-\mathbf{u})) \\
&= (-b\mathbf{u} - a\mathbf{v} - \mathbf{u} \times \mathbf{v}, ba - \mathbf{u} \cdot \mathbf{v}) \\
&= (-a\mathbf{v} - b\mathbf{u} - \mathbf{u} \times \mathbf{v}, ab - \mathbf{u} \cdot \mathbf{v}) \\
&= (\mathbf{p}\mathbf{q})^*
\end{aligned}$$

b)

$$\begin{aligned}(\mathbf{p}+\mathbf{q})^* &= (-\mathbf{u}-\mathbf{v}, a+b)\\ &= (-\mathbf{u}, a)+(-\mathbf{v}, b)\\ &= \mathbf{p}^*+\mathbf{q}^*\end{aligned}$$

c)

$$\begin{aligned}(s\mathbf{q})^* &= (s\mathbf{v}, sb)^*\\ &= (-s\mathbf{v}, sb)\\ &= s(-\mathbf{v}, b)\\ &= s\mathbf{q}^*\end{aligned}$$

d)

$$\begin{aligned}\|\mathbf{pq}\|^2 &= (\mathbf{pq})(\mathbf{pq})^*\\ &= \mathbf{pqq}^*\mathbf{p}^*\\ &= \mathbf{p}\|\mathbf{q}\|^2\mathbf{p}^*\\ &= \mathbf{pp}^*\|\mathbf{q}\|^2\\ &= \|\mathbf{p}\|^2\|\mathbf{q}\|^2\\ \therefore \|\mathbf{pq}\| &= \|\mathbf{p}\|\|\mathbf{q}\|\end{aligned}$$

17. Prove $\mathbf{a}\cdot\frac{\sin\left((1-t)\theta\right)\mathbf{a}+\sin(t\theta)\mathbf{b}}{\sin\theta} = \cos(t\theta)$ algebraically.

Solution:

The key component is to apply the trig identity:

$$\sin\big((1-t)\theta\big) = \sin(\theta - t\theta) = \sin(\theta)\cos(t\theta) - \cos(\theta)\sin(t\theta)$$

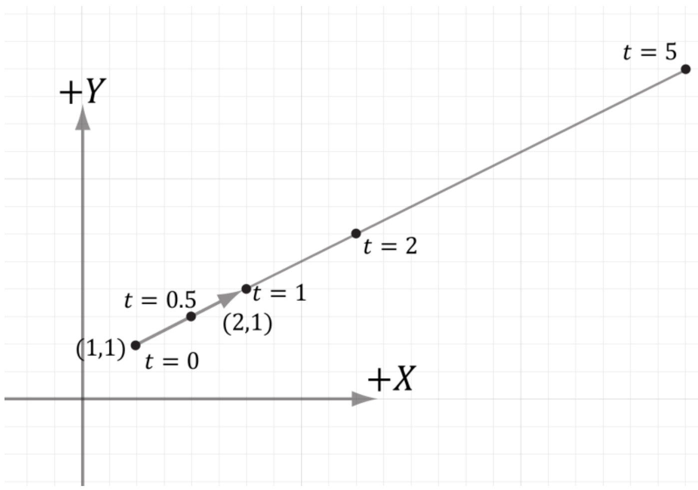
Now,

$$\begin{aligned}&\mathbf{a}\cdot\frac{\sin\big((1-t)\theta\big)\,\mathbf{a}+\sin(t\theta)\,\mathbf{b}}{\sin\theta}\\ &= \frac{\sin\big((1-t)\theta\big)\,\mathbf{a}\cdot\mathbf{a}+\sin(t\theta)\,\mathbf{a}\cdot\mathbf{b}}{\sin\theta}\\ &= \frac{\sin\big((1-t)\theta\big)+\sin(t\theta)\cos\theta}{\sin\theta}\\ &= \frac{\sin(\theta)\cos(t\theta)-\cos(\theta)\sin(t\theta)+\sin(t\theta)\cos\theta}{\sin\theta}\\ &= \frac{\sin(\theta)\cos(t\theta)}{\sin\theta}\\ &= \cos(t\theta)\end{aligned}$$

Appendix C

1. Let $\mathbf{p}(t) = (1,1) + t(2,1)$ be a ray relative to some coordinate system. Plot the points on the ray at *t* = 0.0, 0.5, 1.0, 2.0, and 5.0.

Solution:



[FigD.14: Plotting points on a line]

3. For each part, find the vector line equation of the line passing through the two points.

a) $\mathbf{p}_1 = (2,-1), \mathbf{p}_2 = (4,1)$
b) $\mathbf{p}_1 = (4,-2,1), \mathbf{p}_2 = (2,3,2)$

Solution:

a)

$$\mathbf{p}_2 - \mathbf{p}_1 = (4,1) - (2,-1) = (2,2)$$

$$\begin{aligned}\mathbf{p}(t) &= \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1)\\ &= (2,-1) + t(2,2)\end{aligned}$$

b)

$$\mathbf{p}_2 - \mathbf{p}_1 = (2,3,2) - (4,-2,1) = (-2,5,1)$$

$$\begin{aligned}\mathbf{p}(t) &= \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1)\\ &= (4,-2,1) + t(-2,5,1)\end{aligned}$$

5. Let $\mathbf{L}(t) = (4,2,2) + t(1,1,1)$ be a line. Find the distance from the following points to the line:

$$\mathbf{q} = (0,0,0)$$
$$\mathbf{q} = (4,2,0)$$

$$\mathbf{q} = (0, 2, 2)$$

Solution:

Apply the formula from the previous exercise.

$$\begin{aligned} d &= \frac{\|(\mathbf{q} - \mathbf{p}) \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\left\|\big((0,0,0) - (4,2,2)\big) \times (1,1,1)\right\|}{\|(1,1,1)\|} \\ &= \frac{\|(-4,-2,-2) \times (1,1,1)\|}{\sqrt{3}} \\ &= \frac{\|(0,2,-2)\|}{\sqrt{3}} \\ &= \frac{2\sqrt{6}}{3} \end{aligned}$$

$$\begin{aligned} d &= \frac{\|(\mathbf{q} - \mathbf{p}) \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\left\|\big((4,2,0) - (4,2,2)\big) \times (1,1,1)\right\|}{\|(1,1,1)\|} \\ &= \frac{\|(0,0,-2) \times (1,1,1)\|}{\sqrt{3}} \\ &= \frac{\|(2,-2,0)\|}{\sqrt{3}} \\ &= \frac{2\sqrt{6}}{3} \end{aligned}$$

$$\begin{aligned} d &= \frac{\|(\mathbf{q} - \mathbf{p}) \times \mathbf{u}\|}{\|\mathbf{u}\|} \\ &= \frac{\left\|\big((0,2,2) - (4,2,2)\big) \times (1,1,1)\right\|}{\|(1,1,1)\|} \\ &= \frac{\|(-4,0,0) \times (1,1,1)\|}{\sqrt{3}} \\ &= \frac{\|(0,4,-4)\|}{\sqrt{3}} \\ &= \frac{4\sqrt{6}}{3} \end{aligned}$$

7. Let $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -5\right)$ be a plane. Define the locality of the following points relative to the plane: $\left(3\sqrt{3}, 5\sqrt{3}, 0\right)$, $\left(2\sqrt{3}, \sqrt{3}, 2\sqrt{3}\right)$, and $\left(\sqrt{3}, -\sqrt{3}, 0\right)$.

Solution:

$$\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}-5=0$$

The plane equation is: $\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}-5=0$. Plugging these points into the left-hand side of the equation gives:

$$\frac{3\sqrt{3}}{\sqrt{3}}+\frac{5\sqrt{3}}{\sqrt{3}}+\frac{0}{\sqrt{3}}-5=3 \Rightarrow \text{Front of plane}$$

$$\frac{2\sqrt{3}}{\sqrt{3}}+\frac{\sqrt{3}}{\sqrt{3}}+\frac{2\sqrt{3}}{\sqrt{3}}-5=0 \Rightarrow \text{On plane}$$

$$\frac{\sqrt{3}}{\sqrt{3}}+\frac{-\sqrt{3}}{\sqrt{3}}+\frac{0}{\sqrt{3}}-5=-5 \Rightarrow \text{Behind plane}$$

9. Let $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\frac{5}{\sqrt{2}}\right)$ be a plane. Find the reflection of the point $(0,1,0)$ about the plane.

Solution:

From the previous exercise, we know that $(3,-2,0)$ is a point on the plane.

Because $\mathbf{n}=\left\|\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)\right\|=1$, we have that

$$\begin{aligned}\mathrm{proj}_{\mathbf{n}}(\mathbf{p}-\mathbf{p}_0)&=(\mathbf{n}\cdot\mathbf{p}-\mathbf{n}\cdot\mathbf{p}_0)\mathbf{n}\\&=(\mathbf{n}\cdot\mathbf{p}+d)\mathbf{n}\end{aligned}$$

Now we can apply the formula from §C.4.9:

$$\begin{aligned}\mathbf{q}&=\mathbf{p}-2\,\mathrm{proj}_{\mathbf{n}}(\mathbf{p}-\mathbf{p}_0)\\&=\mathbf{p}-2(\mathbf{n}\cdot\mathbf{p}+d)\mathbf{n}\\&=\mathbf{p}-2\left(\frac{1}{\sqrt{2}}+\frac{5}{\sqrt{2}}\right)\mathbf{n}\\&=(0,1,0)-\frac{12}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)\\&=(0,1,0)+(6,-6,0)\\&=(6,-5,0)\end{aligned}$$

10. Let $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-5\right)$ be a plane, and let $\mathbf{r}(t)=(-1,1,-1)+t(1,0,0)$ be a ray. Find the point at which the ray intersects the plane. Then write a short program using the `XMPlaneIntersectLine` function to verify your answer.

Solution:

$$t_0 = \frac{-\mathbf{n} \cdot \mathbf{p}_0 - d}{\mathbf{n} \cdot \mathbf{u}} = \frac{\frac{1}{\sqrt{3}} + 5}{\frac{1}{\sqrt{3}}} = 1 + 5\sqrt{3}$$

$$\begin{aligned} \mathbf{r}\left(1 + 5\sqrt{3}\right) &= (-1,1,-1) + \left(1 + 5\sqrt{3}\right)(1,0,0) \\ &= (-1,1,-1) + \left(1 + 5\sqrt{3}, 0,0\right) \\ &= \left(5\sqrt{3}, 1, -1\right) \end{aligned}$$

We plug $\left(5\sqrt{3}, 1, -1\right)$ into the plane equation to verify it indeed lies on the plane:

$$\frac{5\sqrt{3}}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{-1}{\sqrt{3}} - 5 = 0 \Rightarrow \text{On plane}$$

The code is given by:

```
#include <windows.h> // for FLOAT definition
#include <DirectXMath.h>
#include <iostream>
using namespace std;

// Overload the  "<<" operators so that we can use cout to
// output XMVECTOR objects.
ostream& operator<<(ostream& os, FXMVECTOR v)
{
	XMFLOAT4 dest;
	XMStoreFloat4(&dest, v);

	os << "(" << dest.x << ", " << dest.y << ", " << dest.z <<
	    ", " << dest.w << ")";
	return os;
}

int main()
{
	XMVECTOR p0 = XMVectorSet(-1.0f, 1.0f, -1.0f, 1.0f);
	XMVECTOR u  = XMVectorSet(1.0f, 0.0f, 0.0f, 0.0f);

	// Construct plane by specifying its (A, B, C, D)
	 // components directly.
	float s = 1.0f / sqrtf(3);
	XMVECTOR plane = XMVectorSet(s, s, s, -5.0f);

	// Function expects a line segment and not a ray; so we just
	// truncate our ray at p0 + 100*u to make a line segment.
	XMVECTOR isect = XMPlaneIntersectLine(plane, p0, p0 + 100*u);

	cout << isect << endl;

	return 0;
}
```

The output is (in homogeneous coordinates so $w = 1$ for points):

```
(8.66025, 1, -1, 1)
Press any key to continue . . .
```

We note $5\sqrt{3} \approx 8.66025$, so the computer result agrees with our calculation.